Econ 704 Macroeconomic Theory Spring 2018*

José-Víctor Ríos-Rull

University of Pennsylvania

Federal Reserve Bank of Minneapolis

CAERP

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*This is the evolution of class notes by many students over the years, both from Penn and Minnesota including Makoto Nakajima (2002), Vivian Zhanwei Yue (2002-3), Ahu Gemici (2003-4), Kagan (Omer Parmaksiz) (2004-5), Thanasis Geromichalos (2005-6), Se Kyu Choi (2006-7), Serdar Ozkan (2007), Ali Shourideh (2008), Manuel Macera (2009), Tayyar Buyukbasaran (2010), Bernabe Lopez-Martin (2011), Rishabh Kirpalani (2012), Zhifeng Cai (2013), Alexandra (Sasha) Solovyeva (2014), Keyvan Eslami (2015), Sumedh Ambokar (2016), Ömer Faruk Koru (2017), Jinfeng Luo (2018), and Ricardo Marto (2019).

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1 Introduction

A model is an artificial economy. The description of a model's environment may include specifying the agents' preferences and endowment, technology available, information structure as well as property rights. The Neoclassical Growth Model is one of the workhorses of modern macroeconomics because it delivers some fundamental properties of industrialized economies, summarized by, among others, Kaldor (1957):

1. Output per capita has grown at a roughly constant rate (2%).
2. The capital-output ratio (where capital is measured using the perpetual inventory method based on past consumption foregone) has remained roughly constant (despite output per capita growth).
3. The capital-labor ratio has grown at a roughly constant rate equal to the growth rate of output.
4. The wage rate has grown at a roughly constant rate equal to the growth rate of output.
5. The real interest rate has been stationary and, during long periods, roughly constant.
6. Labor income as a share of output has remained roughly constant (0.66).
7. Hours worked per capita have been roughly constant.

Equilibrium can be defined as a prediction of what will happen and therefore it is a mapping from environments to outcomes (allocations, prices, etc.). One equilibrium concept that we will deal with is Competitive Equilibrium.[1] Characterizing the equilibrium, however, usually involves finding solutions to a system of an infinite number of equations. There are generally two ways of getting around this. First, invoke the welfare theorem to solve for the allocation and then find the equilibrium prices associated with it. This may sometimes not work due to, say, the presence of externalities. The second way is to resort to dynamic programming and study a Recursive Competitive equilibrium, in which equilibrium objects are functions instead of variables.

[1] Arrow-Debreu or Valuation Equilibrium.

2 Review: Neoclassical Growth Model

We review briefly the basic neoclassical growth model.

2.1 The Neoclassical Growth Model (Without Uncertainty)

The commodity space is

$$\mathcal{L} = \{(l_1, l_2, l_3) : l_i = (l_{it})_{t=0}^{\infty}\ l_{it} \in \mathbb{R},\ \sup_t |l_{it}| < \infty,\ i = 1, 2, 3\}.$$

The consumption possibility set is

$$X(\overline{k}_0) = \{x \in \mathcal{L} :\ \exists (c_t, k_{t+1})_{t=0}^{\infty} \text{ s.th. } \forall t = 0, 1, \ldots$$
$$c_t, k_{t+1} \geq 0,\ x_{1t} + (1-\delta)k_t = c_t + k_{t+1},\ -k_t \leq x_{2t} \leq 0,\ -1 \leq x_{3t} \leq 0,\ k_0 = \overline{k}_0\}.$$

The production possibility set is $Y = \prod_t Y_t$, where

$$Y_t = \{(y_{1t}, y_{2t}, y_{3t}) \in \mathbb{R}^3 :\ 0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})\}.$$

Definition 1 *An Arrow-Debreu equilibrium is* $(x^*, y^*) \in X \times Y$*, and a continuous linear functional* ν^* *such that*

1. $x^* \in \arg\max_{x \in X, \nu^*(x) \leq 0} \sum_{t=0}^{\infty} \beta^t u(c_t(x), -x_{3t})$,
2. $y^* \in \arg\max_{y \in Y} \nu^*(y)$,
3. *and* $x^* = y^*$.

Note that in this definition we have added leisure. Now, let's look at the one-sector growth model's

Social Planner's Problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, -x_{3t}) \qquad (SPP)$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t = x_{1t}$$

$$-k_t \le x_{2t} \le 0$$

$$-1 \le x_{3t} \le 0$$

$$0 \le y_{1t} \le F(-y_{2t}, -y_{3t})$$

$$x = y$$

$$k_0 \text{ given.}$$

Suppose we know that a solution in sequence form exists for (SPP) and is unique.

Exercise 1 *Clearly stating sufficient assumptions on utility and production function, show that (SPP) has a unique solution.*

Two important theorems show the relationship between CE allocations and Pareto optimal allocations:

Theorem 1 (FWT) *Suppose that for all $x \in X$ there exists a sequence $(x_k)_{k=0}^{\infty}$, such that for all $k \ge 0$, $x_k \in X$ and $U(x_k) > U(x)$. If (x^*, y^*, ν^*) is an Arrow-Debreu equilibrium, then (x^*, y^*) is Pareto efficient allocation.*

Theorem 2 (SWT) *If X is convex, preferences are convex, U is continuous, Y is convex and has an interior point, then for any Pareto efficient allocation (x^*, y^*) there exists a continuous linear functional ν such that (x^*, y^*, ν) is a quasiequilibrium, that is:*
(i) for all $x \in X$ such that $U(x) \ge U(x^*)$ it implies $\nu(x) \ge \nu(x^*)$;
(ii) for all $y \in Y$, $\nu(y) \le \nu(y^*)$.

Note that at the very basis of the CE definition and welfare theorems there is an implicit assumption of perfect commitment and perfect enforcement. Note also that the FWT implicitly assumes there is no externality or public goods (achieves this implicit assumption by defining a consumer's utility function only on his own consumption set but no other points in the commodity space). The Greenwald-Stiglitz theorem establishes the Pareto inefficiency of market economies with imperfect information and incomplete markets.

From the First Welfare Theorem, we know that if a Competitive Equilibrium exists, it is Pareto Optimal. Moreover, if the assumptions of the Second Welfare Theorem are satisfied and if the SPP has a unique solution, then the competitive equilibrium allocation is unique and is the same as the PO allocation. Prices can be constructed using this allocation and first-order conditions.

Exercise 2 *Show that*

$$\frac{v_{2t}}{v_{1t}} = F_k(k_t, l_t) \text{ and } \frac{v_{3t}}{v_{1t}} = F_l(k_t, l_t).$$

One shortcoming of the AD equilibrium is that all trade occurs at the beginning of time. This assumption is unrealistic. Modern economics is based on sequential markets. Therefore, we define another equilibrium concept, Sequential Markets Equilibrium (SME). We can easily show that SME is equivalent to ADE by introducing AD securities. All of our results still hold and SME is the right problem to solve.

Exercise 3 *Define a Sequential Markets Equilibrium (SME) for this economy. Prove that the objects we get from the AD equilibrium satisfy SME conditions and that the converse is also true. We should first show that a CE exists and therefore coincides with the unique solution of (SPP).*

Note that the (SPP) problem is hard to solve, since we are dealing with an infinite number of choice variables. We have already established the fact that this SPP problem is equivalent to the following dynamic problem (removing leisure from now on), which is easier to solve:

$$\begin{aligned} v(k) = \max_{c,k'} \quad & u(c) + \beta v(k') \qquad (RSPP) \\ \text{s.t. } & c + k' = f(k). \end{aligned}$$

2.2 A Comment on the Welfare Theorems

Situations in which the welfare theorems would not hold include externalities, public goods, situations in which agents are not price-takers (e.g. monopolies), some legal systems or lacking of markets, which rule out certain contracts that appear to be complete or search frictions. What happens in such situations? The solutions to the Social Planner problem and the CE do not coincide, and so we cannot use the theorems we have developed for dynamic programming. As we will see in this course, we can work with Recursive Competitive Equilibria. In general, we can prove that the solution to the RCE coincides with the one derived from the SME, but not the other way around (for example when we have multiple equilibria). However, in all the models we see in this course, this equivalence will hold.

3 Recursive Competitive Equilibrium

3.1 A Simple Example

We have so far established the equivalence between the allocation of the SP problem, which gives the unique Pareto optimal allocation, and the allocations of the AD equilibrium and the SME. We can now solve for the very complicated equilibrium allocation by solving the relatively easier Dynamic Programming problem of the social planner. One handicap of this approach is that in many environments the equilibrium is not Pareto Optimal and hence not a solution of a social planner's problem (e.g. when taxes are distortionary or when externalities are present). Therefore, the recursive formulation of the problem would not be the right problem to solve. In some of these situations we can still write the problem in sequence form. However, we would lose the powerful computational techniques of dynamic programming. In order to resolve this issue we will define the Recursive Competitive Equilibrium equivalent to SME that we can always solve for.

In order to write the household problem recursively, we need to use equilibrium conditions so that the household knows what prices are, in particular, as a function of some economy-wide aggregate state variables. Let aggregate capital be K and aggregate labor $L = 1$. Then from solving the firm's problem, factor prices are given by $w(K) = F_n(K, 1)$ and $R(K) = F_k(K, 1)$. Therefore, since households take prices as given, they need to know aggregate capital. A household who is deciding about how much to consume and how much to work has to know the whole sequence of future prices in order to make her decision and that means that she needs to know the path of aggregate capital. Therefore, if she believes that aggregate capital changes according to the mapping G such that $K' = G(K)$, then knowing aggregate capital today she would be able to project the path of aggregate capital into the future and thus the path for prices. So, we can write the household problem given function $G(\cdot)$ as follows:

$$
\begin{aligned}
\Omega(K, a; G) = \max_{c,a'} \quad u(c) \quad &+ \quad \beta\Omega(K', a'; G) \qquad\qquad (RCE) \\
\text{s.t. } c + a' \quad &= \quad w(K) + R(K)a \\
K' \quad &= \quad G(K), \\
c \quad &\geq \quad 0
\end{aligned}
$$

The above problem is the problem of a household that sees K in the economy, has a belief G about its evoluation, and carries a units of assets from past. The solution of this problem yields policy functions $c(K, a; G)$ for consumption and $g(K, a; G)$ for next period asset holdings, as well as a value function $\Omega(K, a; G)$. The price functions $w(K), R(K)$ are obtained from the firm's FOCs (below).

$$
\begin{aligned}
u_c(c(K, a; G)) &= \beta\Omega_{a'}(G(K), g(K, a; G); G) \\
\Omega_a(K, a; G) &= R(K)\, u_c(c(K, a; G))
\end{aligned}
$$

Now we can define the Recursive Competitive Equilibrium.

Definition 2 *A Recursive Competitive Equilibrium with arbitrary expectations G is a set of functions Ω, $g : \mathcal{K} \times \mathcal{A} \to \mathbb{R}$, and R, w, $H : \mathcal{K} \to \mathbb{R}_+$ such that:*[2]

[2] Note that we could add the policy function for consumption $c(K, a; G)$.

1. *Given* G, w, R*,* Ω, g *solves the household problem in (RCE),*

2. $K' = H(K;G) = g(K,K;G)$ *(representative agent condition),*

3. $w(K) = F_n(K,1)$ *, and*

4. $R(K) = F_k(K,1)$*.*

Note that G are some arbitrary expectations and do not have to necessarily be rational. Next, we define another notion of equilibrium where the expectations of the households are consistent with what happens in the economy:

Definition 3 *A Rational Expectations Recursive Competitive Equilibrium (REE) is a set of functions* Ω*,* g*,* R*,* w*,* G^**, such that:*

1. *Given* w, R*,* $\Omega(K,a;G^*), g(K,a;G^*)$ *solves HH problem in (RCE),*

2. $K' = G^*(K) = g(K,K;G^*)$*,*

3. $w(K) = F_n(K,1)$*, and*

4. $R(K) = F_k(K,1)$*.*

What this means is that in a REE, households optimize given what they believe is going to happen in the future and what happens in the aggregate is consistent with the household's decision. The proof that every REE can be used to construct a SME is left as an exercise. The reverse turns out not to be true. Notice that in a REE, function G^* projects next period's aggregate capital. In fact, if we construct an equilibrium path based on a REE, once a level of aggreagte capital is reached in some period, then next period aggreagte capital is uniquely pinned down by the transition function G^*. If we have multiplicity of SME, this would imply that we cannot construct the function G^* since one value of capital today could imply more than one value of capital tomorrow. From now on, we will focus on REE unless otherwise stated.

Remark 1 *Note that unless otherwise stated, we will assume that the capital depreciation rate δ is 1, with the firm's profits given by $F(K,1) - \delta K - r(K)K - w(K)$. $R(K)$ is the gross rate of return on capital, which is given by $F_k(K,1) + 1 - \delta$. The net rate of return on capital is $r(K) = F_k(K,1) - \delta$.*

3.2 The Envelope Theorem and the Functional Euler equation

To solve for the RCE and, in particular, to derive the household's optimality conditions we use envelope theorem. This method is valid because of time consistency of consumption choice.

Take the household's problem given by

$$
\begin{aligned}
V(K,a) = \max_{c,a'} \quad u(c) \quad &+ \quad \beta V(K',a') \\
\text{s.t.} \quad c + a' \quad &= \quad w(K) + R(K)a \\
K' \quad &= \quad G(K) \\
c \quad &\geq \quad 0
\end{aligned}
$$

with decision rules for consumption and next period asset holdings given by $c = c(K,a)$ and $a' = g(K,a)$.

By taking the first-order conditions (assuming an interior solution since u is well behaved), we get:

$$-u_c(c) + \beta V_{a'}(K',a') = 0,$$

which evaluated at the optimum is

$$-u_c\left(w(K) + R(K)a - g(K,a)\right) + \beta V_{a'}\left(G(K), g(K,a)\right) = 0 \tag{1}$$

The problem with solving the functional Euler equation is that $V_{a'}$ is not known. However, we can write the value function as a function of current states and differentiate both sides with respect to a.[3] Since the Euler equation holds for all (a, K), we have

$$V(K, a) = u(w(K) + R(K)a - g(K, a)) + \beta V(G(K), g(K, a)) \tag{2}$$

and using the implicit function theorem we can get its derivative with respect to a:

$$\begin{aligned} V_a(K, a) \;\; &= \;\; u_c(w(K) + R(K)a - g(K, a))R(K) + \\ & \quad \frac{\partial g(K, a)}{\partial a} \left[-u_c(w(K) + R(K)a - g(K, a)) + \beta V_{a'}(G(K), g(K, a)) \right] \end{aligned} \tag{3}$$

The term in square brackets in the right hand side is the first-order condition (1) and hence it is zero. So equation (3) simplifies to $V_a(K, a) = u_c(w(K) + R(K)a - g(K, a))R(K)$. Note, however, that we need $V_{a'}(G(K), g(K, a))$ to find the optimal asset holdings allocation. We would need to follow the same procedure for $V(G(K), g(K, a))$, but since equation 1 holds for all (a, K) next period's Euler equation is $u_{c'}(w(G(K)) + R(G(K))g(K, a) - g(G(k), g(K, a))) = \beta V_{a''}\Big[G(G(K)), g(G(K), g(K, a))\Big]$. This in turn implies that $V_{a'}(G(K), g(K, a)) = u_{c'}(w(G(K)) + R(G(K))g(K, a) - g(G(k), g(K, a)))\ R(G(K))$.

Finally, we can replace that in equation (1) and get the functional Euler equation

$$u_c\left(w(K) + R(K)a - g(K, a)\right) - \beta u_{c'}(w(G(K)) + R(G(K))g(K, a) - g(G(k), g(K, a)))\ R(G(K)) = 0 \tag{4}$$

To illustrate this point, consider an individual who wants to loose weight and decides whether to start diet or not. However, he would rather postpone diet for tomorrow and prefer to eat well today. Let 1 denotes that he obeys the diet restrictions and 0 otherwise. Let his preference ordering be given by:

1. $(0, 1, 1, 1...)$
2. $(1, 1, 1, 1...)$

[3] Under some assumptions, V is differentiable. See p. 121 of Prof. Krueger's notes for details.

3. $(0, 0, 0, 0, ...)$

Even though he promises himself that he will start diet tomorrow and chooses to eat well today, tomorrow he will face the same problem. So he will choose the same option again tomorrow. He will thus never start diet and will end up with his least preferred option: $(0, 0, 0, 0, ...)$.

However, in our model that is not what happens. Agents' preferences are time consistent, so what an individual promises today has to be optimal for her tomorrow as well. And that is why we can use the envelope theorem.

3.3 Economies with Government Expenditures

3.3.1 Lump-Sum Tax

The government levies each period T units of goods in a lump sum fashion and spends it in a public good, say, medals. Assume however that consumers do not care about medals. The household's problem is

$$\begin{aligned} V(K, a) = \max_{c, a'} \quad u(c) \quad &+ \quad \beta V(K', a') \\ \text{s.t.} \quad c + a' \quad &= \quad w(K) + R(K)a - T \\ K' \quad &= \quad G(K) \\ c \quad &\geq \quad 0 \end{aligned}$$

Let the solution of this problem be given by policy function $g_a(K, a; M, T)$ and value function $V(K, a; M, T)$. The equilibrium can be characterized by $G^*(K; M, T) = g_a(K, K; G^*, M, T)$ and $M^* = T$ (the government budget constraint is balanced period by period). We will write a complete definition of equilibrium for a version with government debt (below).

Exercise 4 *Define the aggregate resource constraint as* $C + K' + M = f(K, 1)$ *for the planner. Show* *that the equilibrium is optimal when consumers do not care about medals.*

Note that if households cared about medals, then the equilibrium would not necessarily be optimal. The social planner would equate the marginal utility of consumption and of medals, while the agent would not.

3.3.2 Labor Income Tax

We have an economy in which the government levies a tax on labor income in order to purchase medals. Medals are goods that provide utility to the consumers.

$$\begin{aligned}
V(K,a) = \max_{c,a'} \quad u(c,M) \quad &+ \quad \beta V(K',a') \\
\text{s.t.} \quad c + a' \quad &= \quad (1-\tau(K))\, w(K) + R(K)a \\
K' \quad &= \quad G(K) \\
c \quad &\geq \quad 0
\end{aligned}$$

with $M = \tau(K) w\,(K)$.

Since leisure is not valued, the labor decision is trivial. Hence, there is no distortion due to taxes and the CE is Pareto optimal.

Exercise 5 *Is there any change in the above implications of optimality if the tax rate is a function of aggregate capital?*

Exercise 6 *Suppose medals do not provide utility to agents but leisure does. Is the CE optimal now? Is it distorted? What if medals also provide utility?*

3.3.3 Capital Income Tax

Now let us look at an economy in which the government levies tax on capital in order to purchase medals. Medals provide utility to the consumers.

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad u(c,M) \quad &+ \quad \beta V(K',a') \\ \text{s.t.} \quad c + a' \quad &= \quad w(K) + a\left[1 + r(K)\left(1-\tau(K)\right)\right] \\ K' \quad &= \quad G(K) \\ c \quad &\geq \quad 0 \end{aligned}$$

with $M = \tau(K)r(K)K$ and $R(K) = 1 + r(K)$. Now, the First Welfare Theorem is no longer applicable and the CE will therefore not be Pareto optimal anymore (if $\tau(K) > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

Exercise 7 *Derive the first order conditions in the above problem to see the wedge introduced by taxes.*

3.3.4 Taxes and Debt

Assume that the government can now issue debt and use taxes to finance its expenditures. Also assume that agents derive utility from these government expenditures.

A government policy consists of capital taxes, spending (medals) as well as bond issuance. When the aggregate states are K and B, as you will see why, then a government policy (in a recursive world) is

$$\tau(K,B),\ M(K,B) \text{ and } B'(K,B).$$

For now, we shall assume these values are chosen so that the equilibrium exists. In this environment, debt issued is relevant for the household because it permits him to correctly infer the amount of taxes. Therefore the household needs to form expectations about the future level of debt from the government.

The government budget constraint now satisfies (with taxes on labor income):

$$B + M(K,B) = \tau(K,B)R(K)K + q(K,B)B'(K,B)$$

Notice that the household does not care about the composition of his portfolio as long as assets have the same rate of return, which is true because of the no arbitrage condition.

The problem of a household with assets a is given by:

$$\begin{aligned} V(K,B,a) \quad = \quad \max_{c,a'} \quad & u(c, M(K,B)) \quad + \quad \beta V(K',B',a') \\ \text{s.t.} \quad c + a' \; &= \; w(K) + aR(K)\left(1 - \tau(K,B)\right) \\ K' \; &= \; G(K,B) \\ B' \; &= \; H(K,B) \\ c \; &\geq \; 0 \end{aligned}$$

Let $g\,(K,B,a)$ be the policy function associated with this problem. Then, we can define a RCE as follows.

Definition 4 *A* Rational Expectations Recursive Competitive Equilibrium, *given policies $M(K,B)$ and $\tau(K,B)$, is a set of functions V, g, G, H, w, and R, such that*

1. *Given w and R, V and g solve the household's problem,*

2. *Factor prices are paid their marginal productivities*

   $$w(K) = F_2(K,1) \text{ and } R(K) = F_1(K,1),$$

3. *Household wealth = Aggregate wealth*

   $$g(K,B,K + q(K^-,B^-)B) = G(K,B) + q(K,B)H(K,B),$$

4. *No arbitrage condition*

$$\frac{1}{q(K,B)} = [1 - \tau(G(K,B), H(K,B))] \, R(G(K)),$$

5. *Government's budget constraint holds*

$$B + M(K,B) = \tau(K,B)R(K)K + q(K,B)H(K,B),$$

6. *Government debt is bounded; i.e.* $\exists$ *some* $\bar{B}$*, such that for all* $K \in \left[0, \widetilde{k}\right)$ *and* $B \leq \bar{B}$*,* $H(K,B) \leq \bar{B}$*.*

4 Some Other Examples

4.1 A Few Popular Utility Functions

Consider the following three utility forms:

1. $u(c, c^{-})$: this function is called *habit formation* utility function. The utility is increasing in consumption today, but decreasing in the deviations from past consumption (e.g. $u(c, c^{-}) = v(c) - (c - c^{-})^2$). Under habit persistence, an increase in current consumption lowers the marginal utility of consumption in the current period ($u''_{1,1} < 0$) and increases it in the next period ($u''_{1,2} > 0$). Intuitively, the more the agent eats today, the hungrier she will be tomorrow. The aggregate state in this setup is K, while the individual states are a and c^{-}.

   Definition 5 *A Recursive Competitive Equilibrium is a set of functions* V, g, G, w, *and* R, *such that*

   (a) *Given* w *and* R, V *and* g *solve the household's problem,*

(b) *Factor prices are paid their marginal productivities*

$$w(K) = F_2(K, 1) \text{ and } R(K) = F_1(K, 1),$$

(c) *Household wealth = Aggregate wealth*

$$g(K, K, F(G^{-1}(K), 1) - K) = G(K).$$

Exercise 8 *Is the equilibrium optimum in this case?*

2. $u\left(c, C^{-}\right)$: this form is called *catching up with Jones*. There is an externality from aggregate consumption to the agent's payoff. Intuitively, agents care about what their neighbors consume. The aggregate states in this case are K and C^{-}, while c^{-} is no longer an individual state.

   Exercise 9 *How does the agent know C?*

   Exercise 10 *Is the equilibrium optimum in this case?*

3. $u\left(c, C\right)$: this form is called *keeping up with Jones*. The aggregate state is K and C is no longer a predetermined variable.

   Exercise 11 *How does the agent know C?*

   Exercise 12 *Is the equilibrium optimum in this case?*

4.2 An Economy with Capital and Land

Consider an economy with capital and land but without labor. A firm in this economy buys and installs capital, and owns one unit of land that is used in production, according to the technology $F\left(K, L\right)$. In other words, a firm is a "chunk of land of area one" (e.g. farmland), in which it installs its own capital (e.g. tractors). The firm's shares are traded in a stock market, which are bought by households.

A household's problem in this economy is given by:

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad u(c) \quad &+ \quad \beta V(K',a') \\ \text{s.t.} \quad c + P(K)a' \quad &= \quad a\left[D(K)+P(K)\right] \\ K' \quad &= \quad G(K) \end{aligned}$$

where a are shares held by the household, $P(K)$ is their price, and $D(K)$ are dividends per share.

The firm's problem is given by

$$\begin{aligned} \Omega(K,k) = \max_{d,k'} \quad d \quad &+ \quad q(K')\Omega(K',k') \\ \text{s.t.} \quad f(k',1) \quad &= \quad d + k' \\ K' \quad &= \quad G(K) \end{aligned}$$

Ω here is the *value of the firm*, measured in units of output today. The value of the firm tomorrow must be discounted into units of output today, which is done by the discount factor $q\,(K')$. Note that the firm needs to know K', using the aggregate law of motion G to do so.

Definition 6 *A Recursive Competitive Equilibrium consists of functions, V, Ω, h, g, d, q, D, P, and G so that:*

1. *Given prices, V and h solve the household's problem,*
2. *Ω, g, and d solve the firm's problem,*
3. *Representative household holds all shares of the firm*

   $$g(K,1) = 1,$$

4. *The capital of the firm when it is representative must equal the aggregate stock of capital*

   $$h(K,K) = G(K),$$

5. *Value of a representative firm must equal its price and dividends*

$$\Omega(K, K) = D(K) + P(K),$$

6. *The dividends of the representative firm must equal aggregate dividends*

$$d(K, K) = D(K)$$

Exercise 13 *One condition is missing in the definition of the RCE above. Find it!* [Hint: it relates the discount factor of the firm $q(G(K))$ with the price and dividends households receive $(P(K), P(G(K))$, and $D(G(K)))$.]

Exercise 14 *Define the RCE if a were savings paying $R(K)$ as opposed to $shares$ of the firm.*

5 Adding Heterogeneity

In the previous section we looked at situations in which recursive competitive equilibria (RCE) were useful. In particular these were situations in which the welfare theorems failed and so we could not use the standard dynamic programming techniques learned earlier. In this section we look at another way in which RCE are helpful, in particular in models with heterogeneous agents.

5.1 Heterogeneity in Wealth

First, let us consider a model in which we have two types of households that differ only in the amount of wealth they own. Say there are two types of agents, labeled type R (for rich) and P (for poor), of measure μ and $1 - \mu$ respectively. Agents are identical other than their initial wealth position and there is no uncertainty in the model. The problem of an agent with wealth a is given by

$$\begin{aligned}
V(K^R, K^P, a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V(K^{R'}, K^{P'}, a') \\
\text{s.t.} \quad c + a' \quad &= \quad w(\mu K^R + (1-\mu)K^P) + aR(\mu K^R + (1-\mu)K^P) \\
K^{i'} \quad &= \quad G^i(K^R, K^P) \quad \text{for } i = R, P.
\end{aligned}$$

Remark 2 *Note that (in general) the decision rules of the two types of agents are not linear (even though they might be* almost *linear); therefore, we cannot add the two states, K^1 and K^2, to write the problem with one aggregate state, in the recursive form.*

Definition 7 *A Recursive Competitive Equilibrium is a set of functions V, g, w, R, G^1, and G^2 such that that:*

1. *Given prices, V solves the household's functional equation, with g as the associated policy function,*

2. *w and R are the marginal products of labor and capital, respectively (watch out for arguments!),*

3. *Consistency: representative agent conditions are satisfied, i.e.*

$$g(K^R, K^P, K^R) = G^R(K^R, K^P),$$

and

$$g(K^R, K^P, K^P) = G^P(K^R, K^P).$$

Remark 3 *Note that $G^R\left(K^R, K^P\right) = G^P\left(K^P, K^R\right)$ (look at the arguments carefully). Why? (How are rich and poor different?)*

Remark 4 *This is a variation of the simple neoclassical growth model. What does the neoclassical growth model say about inequality? In the steady state, the Euler equations for the two different types* *simplify to*

$$u'\left(c^{R^*}\right) = \beta R u'\left(c^{R^*}\right), \text{ and } u'\left(c^{P^*}\right) = \beta R u'\left(c^{P^*}\right).$$

and we thus have $\beta R = 1$, where

$$R = F_K\left(\mu K^{R^*} + (1-\mu)K^{P^*}, 1\right).$$

Finally, by the household's budget constraint, we must have:

$$c^i + a^i = w + a^i R \quad \text{for } i = R, P$$

where $a^i = K^i$ by the representative agent's condition. Therefore, we have three equations (budget constraints and Euler equation) and four unknowns (a^{i^*} and c^{i^*} for $i = R, P$). This implies that this theory is silent about the distribution of wealth in the steady state!

This is an important implication of the aggregation property. In fact, in the neoclassical growth model with n agents that only differ in their initial endowments, with homothetic preferences, there is a continuum with dimension $n-1$ of steady state wealth distributions.

As we will see throughout the course, heterogeneity will matter in various situations. In the setup we have discussed above, however, wealth heterogeneity did not matter. This aggregation property applied to our macroeconomic context (see Gorman's aggregation theorem for further details) states that if agents' individual savings decision is linear in their individual state (i.e. $g(K,a) = \mu^i(K) + \lambda(K)a$, with $\lambda(K)$ being the marginal propensity to save common to all agents) provided that they all have the same preferences, then aggregate capital can be expressed as the choice of a representative agent (with savings decision given by $g(K,K) = \bar{\mu}(K) + \lambda(K)K$).

Remark 5 *Does this property hold when discount factors or coefficients of relative risk aversion are heterogeneous?*

5.2 Heterogeneity in Skills

Now, consider a slightly different economy in which type i has labor skill ϵ_i. Measures of agents' types, μ^1 and μ^2, satisfy $\mu^1\epsilon_1 + \mu^2\epsilon_2 = 1$ (below we will consider the case in which $\mu^1 = \mu^2 = 1/2$).

The question we have to ask ourselves is: would the value functions of the two types remain the same, as in the previous subsection? The answer turns out to be no!

The problem of the household $i \in \{1, 2\}$ can be written as follows:

$$\begin{aligned} V^i(K^1, K^2, a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V^i(K^{1'}, K^{2'}, a') \\ \text{s.t.} \quad c + a' \quad &= \quad w\left(\frac{K^1 + K^2}{2}\right)\epsilon_i + aR\left(\frac{K^1 + K^2}{2}\right) \\ K^{i'} \quad &= \quad G^i(K^1, K^2) \quad \text{for } i = 1, 2. \end{aligned}$$

Notice that we have indexed the value function by the agent's type and thus the policy function is also indexed by i. The reason is that the marginal product of the labor supplied by each of these types is different (think of $w^i\left(\frac{K^1+K^2}{2}\right) = w\left(\frac{K^1+K^2}{2}\right)\epsilon_i$).

Exercise 15 *Define the RCE.*

Remark 6 *We can also rewrite this problem as*

$$\begin{aligned} V^i(K, \lambda, a) = \max_{c,a'} \quad & \left\{u(c) + \beta V^i(K', \lambda', a')\right\} \\ s.t. \quad & c + a' = R(K)\, a + W(K)\, \epsilon_i \\ & K = G(K, \lambda) \\ & \lambda' = H(K, \lambda), \end{aligned}$$

where K is the total capital in this economy, and λ is the share of one type in total wealth (e.g. type 1).

Then, if g^i is the policy function of type i, then the consistency conditions of the RCE must be:

$$G(K,\lambda) = \frac{1}{2}\left[g^1(K,\lambda,2\lambda K) + g^2(K,\lambda,2(1-\lambda)K)\right],$$

and

$$H(K,\lambda) = \frac{g^1(K,\lambda,2\lambda K)}{2G(K,\lambda)}.$$

5.3 An International Economy Model

In an international economy model the definition of country is an important one. We can introduce the idea of different locations or geography, countries can be victims of different policies, trade across countries maybe more difficult due to different restrictions.

Here we will focus on a model with two countries, 1 and 2, where labor is not mobile between the countries, but capital markets perfect and thus investment can flow freely across countries. However, in order to use it in production, it must have been installed in advanced. Traded goods flow within the period.

The aggregate resource constraint is:

$$C^1 + C^2 + K^{1'} + K^{2'} = F(K^1,1) + F(K^2,1)$$

Suppose that there is a mutual fund that owns the firms in each country and chooses labor in each country and capital to be installed. Its shares are traded in the market and thus, as in the economy with capital and land, individuals own shares of this mutual fund.

The first question to ask, as usual, is *what are the appropriate states in this world?* As it is apparent from the resource constraint and production functions, we need the capital in each country. Moreover, we need to know total wealth in each country. Therefore, we need an additional variable as the aggregate state: the shares owned by country 1 is a sufficient statistic.

We can then write the country i's household problem as:

$$
\begin{aligned}
V^i\left(K^1, K^2, A, a\right) = \max_{c, a'(z)} \quad & u\left(c\right) + \beta V^i\left(K^{1'}, K^{2'}, A', a'\right) \\
s.t. \quad & c + Q(K^1, K^2, A)a' = w^i\left(K^i\right) + a\Phi(K^1, K^2, A) \\
& K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2 \\
& A' = H\left(K^1, K^2, A\right)
\end{aligned}
$$

where A is the total amount of shares in the mutual fund that individuals in country 1 own and a is the amount of shares that an individual owns in country i.

Since labor is immobile and capital is installed in advanced, the wage is country-specific and is simply given by the marginal product of labor: $w^i(K^i) = F_N^i(K^i, 1)$.

Now let's look at the problem of the mutual fund:

$$
\begin{aligned}
\Phi(K^1, K^2, A, k^1, k^2) \quad = \quad & \max_{k^{1'}, k^{2'}, n^1, n^2} \sum_i \left[F^i(k^i, n^i) - n^i w^i(K_i) - k^{i'}\right] + \\
& \qquad \frac{1}{R(K^{1'}, K^{2'}, A)} \Phi(K^{1'}, K^{2'}, A', k^{1'}, k^{2'}) \\
s.t. \qquad & K^{i'} = G^i\left(K^1, K^2, , A\right), \quad \text{for } i = 1, 2 \\
& A' = H\left(K^1, K^2, A\right)
\end{aligned}
$$

Definition 8 *A Recursive Competitive Equilibrium for the (world's) economy is a list of functions,* $\{V^i, h^i, g^i, n^i, w^i, G^i\}_{i=1,2}$*,* Φ*,* H*,* Q*, and* R*, such that the following conditions hold:*

1. *Given prices,* V^i *and* h^i *solve the household's problem in country* i *(for* $i \in \{1, 2\}$*),*
2. *Given prices,* Φ*,* $\{g^i, n^i\}_{i=1,2}$ *solves the mutual fund problem,*

3. *Labor markets clear*

$$n^i(K^1, K^2, A, K^1, K^2) = 1 \qquad \text{for } i = 1, 2,$$

4. *Consistency (MF)*

$$g^i(K^1, K^2, A, K^1, K^2) = G^i(K^1, K^2, A) \qquad \text{for } i = 1, 2,$$

5. *Consistency (Households)*

$$h^1(K^1, K^2, A, A) = H(K^1, K^2, A)$$

and

$$h^1(K^1, K^2, A, A) + h^2(K^1, K^2, A, 1 - A) = 1,$$

6. $Q(K^1, K^2, A) = \frac{1}{R(K^{1'}, K^{2'}, A')}\Phi(K^{1'}, K^{2'}, A')$.

Exercise 16 *Solve for the mutual fund's decision rules. Is next period capital in each country chosen by the mutual fund priced differently? What about labor?*

6 Stochastic Economies

6.1 A Review

6.1.1 Markov Processes

From now on, we will focus on stochastic economies, in which productivity shocks affects the economy. The stochastic process for productivity that we assume is a first-order Markov Process that takes on a finite number of values in the set $Z = \{z^1 < \cdots < z^{n_z}\}$. A first order Markov process implies

$$\Pr(z_{t+1} = z^j | h_t) = \Gamma_{ij}, \qquad z_t(h_t) = z^i$$

where h_t is the history of previous shocks. Γ is a Markov chain with the property that the elements of each rows sum to 1.

Let μ be a probability distribution over initial states, i.e.

$$\sum_i \mu_i = 1$$

and $\mu_i \geq 0 \; \forall i = 1, ..., n_z$.

For next periods the probability distribution can be found by $\mu' = \Gamma^T \mu$.

If Γ is "nice" then $\exists$ a unique μ^* s.t. $\mu^* = \Gamma^T \mu^*$ and $\mu^* = \lim_{m\to\infty}(\Gamma^T)^m \mu_0$, $\forall \mu_0 \in \Delta^i$.

Γ induces the following probability distribution conditional on the initial draw z_0 on $h_t = \{z^0, z^1, ..., z^t\}$:

$\Pi(\{z^0, z_1\}) = \Gamma_{i,.} \qquad$ for $z^0 = z_i$.

$\Pi(\{z^0, z_1, z_2\}) = \Gamma^T \Gamma_{i,.} \qquad$ for $z^0 = z_i$.

Then, $\Pi(h_t)$ is the probability of history h_t conditional on z^0. The expected value of z' is $\sum_{z'} \Gamma_{zz'} z'$ and $\sum_{z'} \Gamma_{zz'} = 1$.

6.1.2 Problem of the Social Planner

Let productivity affect the production function in a multiplicative fashion; i.e. technology is $zF(K, N)$, where z is a shock that follows a Markov chain on a finite state-space. Recall that the problem of the social planner problem (SPP) in sequence form is

$$\max_{\{c_t(z^t),k_{t+1}(z^t)\}\in X(z^t)} \quad \sum_{t=0}^{\infty}\sum_{z^t}\beta^t\pi(z^t)u(c_t(z^t))$$
$$s.t. \quad c_t(z^t)+k_{t+1}(z^t)=z_tF(k_t(z^{t-1}),1),$$

where z_t is the realization of the shock in period t, and z^t is the history of shocks up to (and including) time t. $X(z^t)$ is similar to the consumption possibility set defined earlier but this is after history z^t has occurred and is for consumption and next period capital.

We can then formulate the stochastic SPP in a recursive fashion as

$$V(z_i,K)=\max_{c,K'} \quad \left\{u(c)+\beta\sum_j \Gamma_{ij}V(z_j,K')\right\}$$
$$s.t. \quad c+K'=z_iF(K,1),$$

where Γ is the Markov transition matrix. The solution to this problem gives us a policy function of the form $K'=G(z,K)$.

In a decentralized economy, the Arrow-Debreu equilibrium can be defined by:

$$\max_{\{c_t(z^t),k_{t+1}(z^t),x_{1t}(z^t),x_{2t}(z^t),x_{3t}(z^t)\}\in X(z^t)} \quad \sum_{t=0}^{\infty}\sum_{z^t}\beta^t\pi(z^t)u(c_t(z^t))$$
$$s.t. \quad \sum_{t=0}^{\infty}\sum_{z^t}p_t(z^t).x_t(z^t)\leq 0,$$

where $X(z^t)$ is again a variant of the consumption possibility set after history z^t has occurred. Ignore the overloading of notation. Note that we are assuming the markets are dynamically complete; i.e. there is a complete set of securities for every possible history.

By the same procedure as before, the SME can be written in the following way:

$$\max_{\{c_t(z^t), b_{t+1}(z^t, z_{t+1}), k_{t+1}(z^t)\}} \quad \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi(z^t) u(c_t(z^t))$$

$$\begin{aligned} s.t. \quad & c_t(z^t) + k_{t+1}(z^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) b_{t+1}(z^t, z_{t+1}) \\ & = k_t(z^{t-1}) R_t(z^t) + w_t(z^t) + b_t(z^{t-1}, z_t) \\ & b_{t+1}\left(z^t, z_{t+1}\right) \geq -B. \end{aligned}$$

To replicate the AD equilibrium, we have introduced Arrow securities to allow agents to trade with each other against possible future shocks.

Note that when there is no heterogeneity, there will be no trade in equilibrium, i.e $b_{t+1}(z^t, z_{t+1}) = 0$ for any z^t, z_{t+1}. Moreover, we have two ways of delivering the goods specified in an Arrow security contract: *after production* and *before production*. In an after production setting, the goods will be delivered after production takes place and can only be consumed or saved for the next period. This is the above setting. It is also possible to allow the consumer to rent the Arrow security income as capital to firms, which will correspond to the before production setting.

An important condition that must hold true in the *before production setting* is the no-arbitrage condition:

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1}) = 1$$

Exercise 17 *Every equilibrium achieved in AD settings can also be achieved in a SM setting, by the relation where*

$$q_t(z^t, z_{t+1}) = p_{1t+1}(z^t, z_{t+1}) / p_{1t}(z^t),$$

$$R_t(z^t) = p_{2t}(z^t) / p_{1t}(z^t),$$

and

$$w_t(z^t) = p_{3t}(z^t)/p_{1t}(z^t).$$

Check from the FOC's that the we get the same allocations in the two settings.

Exercise 18 *The problem above assumes state contingent goods are delivered in terms of consumption goods. Instead of this assume they are delivered in terms of capital goods. Show that the same allocation would be achieved in both settings.*

6.1.3 Recursive Competitive Equilibrium

Assume that households can trade state contingent assets, as in the sequential markets case above. Then, we can write a household's problem in recursive form as:

$$\begin{aligned} V(K,z,a) = \max_{c,k',b(z')} \quad & u(c) + \beta \sum_{z'} \Gamma_{zz'} V(K',z',a'(z')) \\ s.t. \quad & c + k' + \sum_{z'} q_{z'}(K,z)\, b(z') = w(K,z) + aR(K,z) \\ & K' = G(K,z) \\ & a'(z') = k' + b(z'). \end{aligned}$$

Exercise 19 *Write the FOC's for this problem, given prices and the law of motion for aggregate capital.*

Definition 9 *A Recursive Competitive Equilibrium is a collection of functions V, k', d, G, w, and R, so that*

1. *Given G, w, and R, V solves the household's functional equation, with k' and b as the associated policy function,*

2. $b(K,z,K;z') = 0$, *for all z',*

3. $k'(K,z,K) = G(K,z)$,

4. $w(K,z) = zF_n(K,1)$ *and* $R(K,z) = zF_k(K,1)$,

5. *and* $\sum_{z'} q_{z'}(K,z) = 1$.

The last condition is known as the no-arbitrage condition (recall that we had this equation in the case of sequential markets as well). To see why this is a necessary equation in the equilibrium, note that an agent can either save in the form of capital or through Arrow securities. However, these two choices must cost the same, which implies Condition 5 above.

Remark 7 *Note that in the SME version of the household problem, in order for households not to achieve infinite consumption, we need a no-Ponzi condition. Such condition is*

$$\lim_{t\to\infty} \frac{a_t}{\prod_{s=0}^{t} R_s} < \infty.$$

This is the weakest condition that imposes no restrictions on the first order conditions of the household's problem. It is harder to come up with its analogue for the recursive case. One possibility is to assume that a' lies in a compact set $\mathcal{A}$, or a set that is bounded from below.[4]

6.2 An International Economy Model with Shocks

We revisit the international economy model studied before and we now add country-specific shocks. Let z_1 and z_2 represent productivity shocks in country 1 and 2, respectively. The aggreagte state variables are now the productivity shocks, the aggregate stocks of capital in each country, and the amount of shares owned by country 1 in the mutual fund.

[4] We must specify $\mathcal{A}$ such that the borrowing constraint implicit in $\mathcal{A}$ is never binding.

The problem of an household in country i is:

$$\begin{aligned}
V^i\left(z_1, z_2, K_1, K_2, A, a\right) = \max_{c, a'(\vec{z}')} \quad & u\left(c\right) + \beta \sum_{\vec{z}'} \Gamma_{\vec{z}\vec{z}'} V^i\left(\vec{z}', \vec{K}', A'(\vec{z}'), a'(\vec{z}')\right) \\
s.t. \quad & c + \sum_{\vec{z}'} q(\vec{z}, \vec{K}, A, \vec{z}')a'(\vec{z}') = w^i\left(z_i, K_i\right) + a\Phi(\vec{z}, \vec{K}, A) \\
& K_i' = G_i\left(\vec{z}, \vec{K}, A\right), \quad \text{for } i = 1, 2 \\
& A(\vec{z}') = H\left(\vec{z}, \vec{K}, A, \vec{z}'\right) \quad \forall \vec{z}'
\end{aligned}$$

Let decision rule for next period asset holdings be $a'(\vec{z}') = h(\vec{z}, \vec{K}, A, a, \vec{z}') \quad \forall \vec{z}'$. Note the financial market structure assumed here. As before, labor is immobile and thus wages are country-specific and given by $w^i(z_i, K_i) = z_i F_N(K_i, 1)$.

Exercise 20 *Write this economy with state-contingent claims in own country only.*

Exercise 21 *Write this economy where individuals can move freely in advance, but with incomplete markets.*

Now let's look at the net present value of the mutual fund in equilibrium:

$$\begin{aligned}
\Phi(\vec{z}, \vec{K}, A) \;=\; & \sum_{z_i} \left[z_i F(K_i, 1) - w^i(z_i, K_i)\right] - \sum_i G_i(\vec{z}, \vec{K}, A) + \\
& \sum_{\vec{z}'} \Gamma_{\vec{z}\vec{z}'} Q(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A, \vec{z}'))\Phi(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A, \vec{z}')) \quad (5)
\end{aligned}$$

where Q represents intertemporal prices, which in equilibrium should satisfy $\forall \vec{z}'$:

$$q(\vec{z}, \vec{K}, A, \vec{z}') = Q(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A, \vec{z}'))\Phi(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A, \vec{z}'))$$

Exercise 22 *There is one more condition for G_i that equates expected return in each country. What is it?*

Definition 10 *A Recursive Competitive Equilibrium for the (world's) economy is a set of functions V^i, h^i, w^i, G_i for $i \in \{1,2\}$, and q, H, Q, and Φ such that the following conditions hold:*

1. *Given prices and laws of motion, V^i and h^i solve the household's problem in country i for $i \in \{1,2\}$,*
2. *The representative agent condition must hold:* $h^1\left(\vec{z}, \vec{K}, A, A, \vec{z}'\right) = H\left(\vec{z}, \vec{K}, A, A, \vec{z}'\right) \quad \forall \vec{z}'$,
3. *The sum of shares in the mutual fund must be 1:* $h^1\left(\vec{z}, \vec{K}, A, A, \vec{z}'\right) + h^2\left(\vec{z}, \vec{K}, A, 1-A, \vec{z}'\right) = 1 \quad \forall \vec{z}'$,
4. *The mutual fund's value Φ satisfies equation 5*
5. *$w^i(z_i, K_i)$ is equated to the marginal products of labor in each country i for $i \in \{1,2\}$,*
6. *Expected rate of return on capital is the same across countries,*
7. $q(\vec{z}, \vec{K}, A, \vec{z}') = Q(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A, \vec{z}'))\Phi(\vec{z}', G(\vec{z}, \vec{K}, A), H(\vec{z}, \vec{K}, A, \vec{z}')) \quad \forall \vec{z}'$,
8. *The aggregate resource constraint must hold:*

$$\sum_i \Bigg[z_i F(K_i, 1) - G_i(\vec{z}, \vec{K}, A) - \Big(w^i(z_i, K_i) + A_i \Phi(\vec{z}, \vec{K}, A) - \sum_{\vec{z}'} q(\vec{z}, \vec{K}, A, \vec{z}') h^i\left(\vec{z}, \vec{K}, A, A_i, \vec{z}'\right)\Big)\Bigg] = 0$$

where $A_1 = A$ and $A_2 = 1 - A$.

6.3 Heterogeneity in Wealth and Skills with Complete Markets

Now, let us consider a model in which we have two types of households, with equal measure $\mu_i = 1/2$, that care about leisure, but differ in the amount of wealth they own as well as their labor skill. There is also uncertainty and Arrow securities like we have seen before.

Let A^1 and A^2 be the aggregate asset holdings of the two types of agents. These will now be state variables for the same reason K^1 and K^2 were state variables earlier. The problem of an agent $i \in \{1, 2\}$ with wealth a is given by

$$
\begin{aligned}
V^i\left(z, A^1, A^2, a\right) = \max_{c,n,a'(z')} \quad & u\left(c, n\right) + \beta \sum_{z'} \Gamma_{zz'} V^i\left(z', A^{1'}(z'), A^{2'}(z'), a'(z')\right) \\
s.t. \quad & c + \sum_{z'} q\left(z, A^1, A^2, z'\right) a'\left(z'\right) = R\left(z, K, N\right) a + W\left(z, K, N\right) \epsilon_i n \\
& A^{i'}(z') = G^i\left(z, A^1, A^2, z'\right), \quad \text{for } i = 1, 2, \forall z' \\
& N = H\left(z, A^1, A^2\right) \\
& K = \frac{A^1 + A^2}{2}.
\end{aligned}
$$

Let $g^i(z, A^1, A^2, a^i)$ and $h^i(z, A^1, A^2, a^i)$ be the asset and labor policy functions be the solution of each type i to this problem. Then, we can define the RCE as below.

Definition 11 *A* Recursive Competitive Equilibrium with Complete Markets *is a set of functions V^i, g^i, h^i, G^i for $i \in \{1, 2\}$, R, w, H, and q, such that:*

1. *Given prices and laws of motion, V^i, g^i and h^i solve the problem of household i for $i \in \{1, 2\}$,*
2. *Labor markets clear:*
   $H\left(z, A^1, A^2\right) = \epsilon_1 h^1\left(z, A^1, A^2, A^1\right) + \epsilon_2 h^2\left(z, A^1, A^2, A^2\right)$,
3. *The representative agent condition:*
   $G^i\left(z, A^1, A^2, z'\right) = g^i\left(z, A^1, A^2, A^i, z'\right) \quad$ *for* $i = 1, 2, \forall z'$
4. *The average price of the Arrow security must satisfy:*
   $\sum_{z'} q\left(z, A^1, A^2, z'\right) = 1$,
5. *$G^1\left(z, A^1, A^2, z'\right) + G^2\left(z, A^1, A^2, z'\right)$ is independent of z' (due to market clearing).*

6. *R and W are the marginal products of capital and labor.*

Exercise 23 *Write down the household problem and the definition of RCE with non-contingent claims instead of complete markets.*

7 Asset Pricing: Lucas Tree Model

We now turn to the simplest of all models in term of allocations as they are completely exogenous, called the *Lucas tree model*. We want to characterize the properties of prices that are capable of inducing households to consume the stochastic endowment.

7.1 The Lucas Tree with Random Endowments

Consider an economy in which the only asset is a tree that gives fruit. The agent's problem is to choose consumption c and the amount of shares of the tree to hold s' according to

$$\begin{aligned} V(z,s) = \max_{c,s'} \quad & u(c) + \beta \sum_{z'} \Gamma_{zz'} V(z',s') \\ s.t. \quad & c + p(z)\, s' = s\left[p(z) + d(z)\right], \end{aligned}$$

where $p(z)$ is the price of the shares (to the tree), in state z, and $d(z)$ is the dividend associated with state z.

Definition 12 *A Rational Expectations Recursive Competitive Equilibrium is a set of functions, V, g, d, and p, such that*

1. *V and g solves the household's problem given prices,*

2. *$d(z) = z$, and,*

3. $g(z,1)=1$, *for all* z.

To explore the problem further, note that the FOC for the household's problem imply the equilibrium condition

$$u_c\left(c\left(z,1\right)\right)=\beta\sum_{z'}\Gamma_{zz'}\left[\frac{p\left(z'\right)+d\left(z'\right)}{p\left(z\right)}\right]u_c\left(c\left(z',1\right)\right).$$

where we have $u_c\left(z\right):=u_c\left(c\left(z,1\right)\right)$. Then this simplifies to

$$p\left(z\right)u_c\left(z\right)=\beta\sum_{z'}\Gamma_{zz'}u_c\left(z'\right)\left[p\left(z'\right)+z'\right]\qquad\forall z.$$

Exercise 24 *Derive the Euler equation for household's problem to show the result above.*

Note that this is just a system of n_z equations with unknowns $\left\{p\left(z_i\right)\right\}_{i=1}^{n}$. We can use the power of matrix algebra to solve the system. To do so, let:

$$\mathbf{p}:=\begin{bmatrix}p\left(z_1\right)\\ \vdots\\ p\left(z_n\right)\end{bmatrix}_{(n_z\times 1)},$$

and

$$\mathbf{u}_c:=\begin{bmatrix}u_c\left(z_1\right) & & 0\\ & \ddots & \\ 0 & & u_c\left(z_n\right)\end{bmatrix}_{(n_z\times n_z)}.$$

Then

$$\mathbf{u}_c.\mathbf{p}=\begin{bmatrix}p\left(z_1\right)u_c\left(z_1\right)\\ \vdots\\ p\left(z_n\right)u_c\left(z_n\right)\end{bmatrix}_{(n_z\times 1)},$$

and

$$\mathbf{u}_c.\mathbf{z} = \begin{bmatrix} z_1 u_c(z_1) \\ \vdots \\ z_n u_c(z_n) \end{bmatrix}_{(n_z \times 1)},$$

Now, rewrite the system above as

$$\mathbf{u}_c \mathbf{p} = \beta \Gamma \mathbf{u}_c \mathbf{z} + \beta \Gamma \mathbf{u}_c \mathbf{p},$$

where Γ is the transition matrix for z, as before. Hence, the price for the shares is given by

$$(\mathbf{I}_{\mathbf{n_z}} - \beta \Gamma) \mathbf{u}_c \mathbf{p} = \beta \Gamma \mathbf{u}_c \mathbf{z},$$

or

$$\mathbf{p} = ([\mathbf{I}_{\mathbf{n_z}} - \beta \Gamma] \mathbf{u}_c)^{-1} \beta \Gamma \mathbf{u}_c \mathbf{z},$$

where $\mathbf{p}$ is the vector of prices that clears the market.

Exercise 25 *How are prices defined when the agent faces taste shocks?*

7.2 Asset Pricing

Consider our simple model of Lucas tree with fluctuating output. What is the definition of an asset in this economy? It is "a claim to a chunk of fruit, sometime in the future."

If an asset, a, promises an amount of fruit equal to $a_t(z^t)$ after history $z^t = (z_0, z_1, \ldots, z_t)$ of shocks, after a set of (possible) histories in H, the price of such an entitlement in date $t = 0$ is given by:

$$p(a) = \sum_t \sum_{z^t \in H} q_t^0(z^t)\, a_t(z^t),$$

where $q_t^0(z^t)$ is the price of one unit of fruit after history z^t in today's "dollars"; this follows from a no-arbitrage argument. If we have the date $t = 0$ prices, $\{q_t\}$, as functions of histories, we can *replicate any possible asset by a set of state-contingent claims* and use this formula to price that asset.

To see how we can find prices at date $t = 0$, consider a world in which the agent wants to solve

$$\begin{aligned}
\max_{c_t(z^t)} \quad & \sum_{t=0}^{\infty} \beta^t \sum_{z^t} \pi_t\left(z^t\right) u\left(c_t\left(z^t\right)\right) \\
s.t. \quad & \sum_{t=0}^{\infty} \sum_{z^t} q_t^0\left(z^t\right) c_t\left(z^t\right) \leq \sum_{t=0}^{\infty} \sum_{h^t} q_t^0\left(z^t\right) z_t.
\end{aligned}$$

This is the familiar Arrow-Debreu market structure, where the household owns a tree, and the tree yields $z \in Z$ amount of fruit in each period. The FOC for this problem imply:

$$q_t^0\left(z^t\right) = \beta^t \pi_t\left(z^t\right) \frac{u_c\left(z_t\right)}{u_c\left(z_0\right)}.$$

This enables us to price the good in each history of the world and price any asset accordingly.

Comment 1 *What happens if we add state-contingent shares b into our recursive model? Then the agent's problem becomes:*

$$\begin{aligned}
V\left(z, s, b\right) = \max_{c, s', b'(z')} \quad & u\left(c\right) + \beta \sum_{z'} \Gamma_{zz'} V\left(z', s', b'\left(z'\right)\right) \\
s.t. \quad & c + p\left(z\right) s' + \sum_{z'} q\left(z, z'\right) b'\left(z'\right) = s\left[p\left(z\right) + z\right] + b.
\end{aligned}$$

A characterization of q can be obtained by the FOC, evaluated at the equilibrium, and thus written as:

$$q\left(z, z'\right) u_c\left(z\right) = \beta \Gamma_{zz'} u_c\left(z'\right).$$

We can thus price all types *of securities using p and q in this economy.*

To see how we can price an asset given today's shock is z, consider *the option to sell it tomorrow* at price P as an example. The price of such an asset today is

$$\hat{q}(z, P) = \sum_{z'} q(z, z') \max\{P - p(z'), 0\},$$

where the agent has the option not to sell it. The American option to sell at price P either tomorrow or the day after tomorrow is priced as:

$$\widetilde{q}(z, P) = \sum_{z'} q(z, z') \max\{P - p(z'), \hat{q}(z', P)\}.$$

Similarly, an European option to buy the asset at price P the day after tomorrow is priced as:

$$\bar{q}(z, P) = \sum_{z'} \sum_{z''} \max\{p(z'') - P, 0\} q(z', z'') q(z, z').$$

Note that $R(z) = \left[\sum_{z'} q(z, z')\right]^{-1}$ is the gross risk free rate, given today's shock is z. The unconditional gross risk free rate is then given by $R^f = \sum_z \mu_z^* R(z)$ where μ^* is the steady-state distribution of the shocks.

The average gross rate of return on the stock market is $\sum_z \mu_z^* \sum_{z'} \Gamma_{zz'} \left[\frac{p(z')+z'}{p(z)}\right]$ and the risk premium is the difference between this rate and the unconditional gross risk free rate (i.e. given by $\sum_z \mu_z^* \left(\sum_{z'} \Gamma_{zz'} \left[\frac{p(z')+z'}{p(z)}\right] - R(z)\right)$).

Exercise 26 *Use the expressions for p and q and the properties of the utility function to show that risk premium is positive.*

7.3 Taste Shocks

Consider an economy in which the only asset is a tree that gives fruits. The fruit is constant over time (normalized to 1) but the agent is subject to preference shocks for the fruit each period given by

$\theta \in \Theta$. The agent's problem in this economy is

$$V(\theta, s) = \max_{c, s'} \quad \theta u(c) + \beta \sum_{\theta'} \Gamma_{\theta\theta'} V(\theta', s')$$
$$s.t. \quad c + p(\theta) s' = s[p(\theta) + d(\theta)].$$

The equilibrium is defined as before. The only difference is that, now, we must have $d(\theta) = 1$ since z is normalized to 1. What does it mean that the output of the economy is constant (fixed at one), but the tastes for this output change? In this setting, the function of the price is to convince agents to keep their consumption constant even in the presence of taste shocks. All the analysis follows through as before once we write the FOC's characterizing the prices of shares, $p(\theta)$, and state-contingent prices $q(\theta, \theta')$.

This is a simple model, in the sense that the household does not have a real choice regarding consumption and savings. Due to market clearing, household consumes what nature provides her. In each period, according to the state of productivity z and taste θ, prices adjusts such that household would like to consume z, which is the amount of fruit that the nature provides. In this setup, output is equal to z. If we look at the business cycle in this economy, the only source of output fluctuations is caused by nature. Eveyrhing determined by the supply side of the economy and the demand side has indeed no impact on output.

In next section, we are going to introduce search frictions to incorporate a role for the demand side into our model.

8 Endogenous Productivity in a Product Search Model

We will model the situation in which households need to find the fruit before consuming it.[5] Assume that households have to find the tree in order to consume the fruit. Finding trees is characterized by

[5] Think of fields in *The Land of Apples*, full of apples, that ,are owned by firms; agents have to buy the apples. In addition, they have to search for them as well!

a constant returns to scale (increasing in both arguments) matching function $M(T, D)$,[6] where T is the *number of trees* in the economy and D is the aggregate *shopping effort* exerted by households when searching. The probability that a tree finds a shopper is given by $\frac{M(T,D)}{T}$, i.e. the total number of matches divided by the number of trees. The probability that a unit of shopping effort finds a tree is given by $\frac{M(T,D)}{D}$, i.e. the total number of matches divided by the economy's effort level.

Let's assume that $M(T, D)$ takes the form $D^{\varphi}T^{1-\varphi}$ and denote $\frac{1}{Q} := \frac{D}{T}$, i.e. the ratio of shoppers per trees, as capturing *the market tightness* (and thus $Q = \frac{T}{D}$). The probability of a household finding a tree is given by $\Psi^h(Q) := \frac{M(T,D)}{D} = Q^{1-\varphi}$ and thus the higher the number of people searching, the smaller the probability of a household finding a tree. The probability of a tree finding a household is then given by $\Psi^f(Q) := \frac{M(T,D)}{T} = Q^{-\varphi}$, and thus the higher the number of people searching, the higher the probability of a tree finding a shopper. Note that in this economy the number of trees is constant and equal to one.[7]

Let us assume households face a demand side shock θ and a supply side shock z. They are follow independent Markov processes with transitional probabilities $\Gamma_{\theta\theta'}$ and $\Gamma_{zz'}$, respectively. Households choose the consumption level c, the search effort exerted to get the fruit d, and the shares of the tree to hold next period s'. The household's problem can be written as

$$V(\theta, z, s) = \max_{c,d,s'} \quad u(c, d, \theta) + \beta \sum_{\theta',z'} \Gamma_{\theta\theta'}\Gamma_{zz'}V(\theta', z', s') \tag{6}$$

$$s.t. \quad c + P(\theta, z)\, s' = P(\theta, z)\left[s\left(1 + \hat{R}(\theta, z)\right)\right] \tag{7}$$

$$c = d\, \Psi^h(Q(\theta, z))\, z. \tag{8}$$

[6] What does the fact that M is constant returns to scale imply?

[7] It is easy to find the statements for Ψ^h and Ψ^f, given the Cobb-Douglas matching function:

$$\Psi^h(Q) = \frac{D^{\varphi}T^{1-\varphi}}{D} = \left(\frac{T}{D}\right)^{1-\varphi} = Q^{1-\varphi},$$
$$\Psi^f(Q) = \frac{D^{\varphi}T^{1-\varphi}}{T} = \left(\frac{T}{D}\right)^{-\varphi} = Q^{-\varphi}.$$

The question is: is Cobb-Douglas an appropriate choice for the matching function, or its choice is a matter of simplicity?

where P is the price of the tree relative to that of consumption and $\hat{R}$ is the dividend income (in units of the tree). Note that the equation 7 is our standard budget constraint, while equation 8 corresponds to the shopping constraint.

Note some notation conventions here. $P(\theta, z)$ is in terms of consumption goods, while $\hat{R}(\theta, z)$ is in terms of shares of the tree (that's why we are using the hat). We could also write the household budget constraint in terms of the price of consumption relative to that of the tree. To do so, let's define $\hat{P}(\theta, z) = \frac{1}{P(\theta, z)}$ as the price of consumption goods in terms of the tree. Then the budget constraint can be defined as:

$$c\hat{P}(\theta, z) + s' = s\left(1 + \hat{R}(\theta, z)\right)$$

Let's maintain our notation with $P(\theta, z)$ and $\hat{R}(\theta, z)$ from now on. We can substitute the constraints into the objective and solve for d in order to get the Euler equation for the household. Using the market clearing condition in equilibrium, the problem reduces to one equation and two unknowns, $P(\theta, z)$ and $Q(\theta, z)$ (other objects, C,D and $\hat{R}$ are known functions of P and Q, and the amount shares of the tree in equilibrium is 1 as before). We thus still need another functional equation to sovel for the equilibrium of this economy, i.e. we need to specify the search protocol. We now turn to one way of doing so.

Exercise 27 *Derive the Euler equation of the household from the problem defined above.*

8.1 Competitive Search

Competitive search is a particular search protocol of what is called non-random (or directed) search. To understand this protocol, consider a world consisting of a large number of islands. Each island has a sign that displays two numbers, $P(\theta, z)$ and $Q(\theta, z)$. $P(\theta, z)$ is the price on the island and $Q(\theta, z)$ is a measure of market tightness in that island (or if the price is a wage rate W, then Q is the number of workers on the island divided by the number of job opportunities in that island). Both individuals and firms have to decide to go to one island. For instance, in an island with a higher wage, the worker might have a higher income conditional on finding a job. However, the probability of finding a job might be low on that island given the tightness of the labor market on that island. The same story holds for the job owners, who are searching to hire workers.

In our economy, both firms and workers search for specific markets indexed by price P and a market tightness Q.[8] An island, or a pair of (P, Q), is operational if there exists some consumer and firm choosing that market. Therefore, an agent should choose P and Q such that it gives sufficient profit to the firm, so that it wants to be in that island as opposed to doing something else, which will be determined in the equilibrium. Competitive search is magic in the sense that it does not presuppose a particular pricing protocol that other search protocols need (e.g. bargaining).

Maintaining the demand shock θ and supply side shock z we introduced before, we can then define the household problem with competitive search as follows

$$V(\theta, z, s) = \max_{c,d,s',P,Q} \quad u(c, d, \theta) + \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V(\theta', z', s') \tag{9}$$

$$s.t. \quad c + Ps' = P\left[s\left(1 + \hat{R}(\theta, z)\right)\right], \tag{10}$$

$$c = d\, \Psi^h(Q)\, z \tag{11}$$

$$\frac{z\Psi^f(Q)}{P} \geq \hat{R}(\theta, z) \tag{12}$$

Let $u(c, d, \theta) = u(\theta c, d)$ from here on. The first two constraints were defined above, while the last is the firm's participation constraint, which is the condition that states that firms would prefer this market to other markets in which they would get $\hat{R}(\theta, z)$.

To solve the problem, let's take the first order conditions. One way to do this is to first plug the first two constraints into the objective function (expressing c and s' as functions of d) and then take the

[8] From now on, we will drop the arguments of P and Q.

derivative with respect to d (recall that $\Psi^h = Q^{1-\varphi}$) to get:

$$\theta Q^{1-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) + u_d(\theta d Q^{1-\varphi} z, d) = \\ \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3\left(\theta', z', s(1+\hat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \frac{Q^{1-\varphi} z}{P} \quad (13)$$

To find V_3 consider the original problem where constraints are not plugged into the objective function. Using the envelope theorem we get:

$$V_3(\theta, z, s) = \left[\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z}\right] P(1+\hat{R}(\theta, z))$$

Combining these two gives the Euler equation:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = \\ \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\hat{R}(\theta', z'))}{P} \left[\theta' u_c(\theta' d' Q'^{1-\varphi} z', d') + \frac{u_d(\theta' d' Q'^{1-\varphi} z', d')}{Q'^{1-\varphi} z'}\right] \quad (14)$$

Observe that this equation is the same as the Euler equation from the random search model. This gives us the optimal search and saving behavior for a given island (i.e. a market tightness $1/Q$ and price level P). To understand which market to search in, we need to look at the FOC with respect to Q and P. Let λ denote the Lagrange multiplier on the firm's participation constraint, then the FOC with respect to Q and P are respectively:

$$\theta d(1-\varphi) Q^{-\varphi} z u_c(\theta d Q^{1-\varphi} z, d) = \\ \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3\left(\theta', z', s(1+\hat{R}(\theta, z)) - \frac{dQ^{1-\varphi} z}{P}\right) \frac{d(1-\varphi) Q^{-\varphi} z}{P} - \lambda \frac{\varphi Q^{-\varphi-1} z}{P} \quad (15)$$

and

$$\beta \sum_{\theta',z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\hat{R}(\theta,z)) - \frac{dQ^{1-\varphi} z}{P} \right) dQ = -\lambda \tag{16}$$

Combining these two equation gives us:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta',z'} \Gamma_{\theta\theta'} \Gamma_{zz'} V_3 \left(\theta', z', s(1+\hat{R}(\theta,z)) - \frac{dQ^{1-\varphi} z}{P} \right) \left[\frac{1}{(1-\varphi)P} \right] \tag{17}$$

Recall that we had defined $V_3(\cdot,\cdot,\cdot)$ above and thus this Euler equation simplifies to

$$(1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) = \beta \sum_{\theta',z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1+\hat{R}(\theta',z'))}{P} \left[\theta' u_c(\theta' d' {Q'}^{1-\varphi} z', d') + \frac{u_d(\theta' d' {Q'}^{1-\varphi} z', d')}{{Q'}^{1-\varphi} z'} \right] \tag{18}$$

Or by equations (14) and (18), we get:

$$\theta u_c(\theta d Q^{1-\varphi} z, d) + \frac{u_d(\theta d Q^{1-\varphi} z, d)}{Q^{1-\varphi} z} = (1-\varphi)\theta u_c(\theta d Q^{1-\varphi} z, d) \tag{19}$$

Now we can define the equilibrium:

Definition 13 *An equilibrium with competitive search consists of functions V, c, d, s', P, Q, and R that satisfy:*

1. *Household's budget constraint, (condition 10)*
2. *Household's shopping constraint, (condition 11)*
3. *Household's Euler equation, (condition 14)*
4. *Market condition, (condition 18)*

5. *Firm's participation constraint, (condition 12), which gives us that the dividend payment is the profit of the firm,* $\hat{R}(\theta, z) = \frac{zQ^{-\varphi}}{P}$*,*

6. *Market clearing, i.e.* $s' = 1$ *and* $Q = 1/d$.

Note that if you had solved the problem by replacing c and d as functions of s', then the Euler equations (14) and (18) would be given by:

$$\theta u_c + \frac{u_d}{Q^{1-\varphi} z} = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P'(1 + \hat{R}(\theta', z'))}{P} \left[\theta' u_c' + \frac{u_d'}{Q'^{1-\varphi} z'} \right] \tag{20}$$

and

$$\theta u_c + \frac{u_d}{Q^{1-\varphi} z} = -\frac{(1-\varphi)}{\varphi} \frac{u_d}{Q^{-1-\varphi} z} \tag{21}$$

where now $u_c = u_c \left(\theta P \left[s \left(1 + \hat{R} \right) - s' \right], \frac{P\left[s\left(1+\hat{R}\right) - s' \right]}{Q^{1-\varphi} z} \right)$ and $u_d = u_d \left(\theta P \left[s \left(1 + \hat{R} \right) - s' \right], \frac{P\left[s\left(1+\hat{R}\right) - s' \right]}{Q^{1-\varphi} z} \right)$.

Also, if the agent's budget constraint would be defined as $c + P(\theta, z)s' = s\left(P(\theta, z) + R\left(\theta, z\right)\right)$, then the firm's participation constraint is given by $z\Psi^f\left(Q(\theta, z)\right) \geq R(\theta, z)$ and the equilibrium conditions are

$$\theta u_c + \frac{u_d}{Q^{1-\varphi} z} = \beta \sum_{\theta', z'} \Gamma_{\theta\theta'} \Gamma_{zz'} \frac{P' + R'}{P} \left[\theta' u_c' + \frac{u_d'}{Q'^{1-\varphi} z'} \right] \tag{22}$$

and

$$\left(\theta u_c + \frac{u_d}{Q^{1-\varphi} z} \right) \left[s \left(1 - \varphi \frac{R}{Q} \right) - s' \right] = (1 - \varphi) Q^{-\varphi} \left(\frac{s\left[P + R\right] - Ps'}{z} \right) u_d \tag{23}$$

where now $u_c = u_c \left(\theta \left[s\left(P + R\right) - Ps' \right], \frac{s[P+R] - Ps'}{Q^{1-\varphi} z} \right)$ and $u_d = u_d \left(\theta \left[s\left(P + R\right) - Ps' \right], \frac{s[P+R] - Ps'}{Q^{1-\varphi} z} \right)$.

Exercise 28 *Define the recursive equilibrium with competitive search for this last setup.*

8.1.1 Firms' Problem

Note that in any given period a firm maximizes its returns to the tree by choosing the appropriate market, Q. Note that, by choosing a market Q, the firm is effectively choosing a price. Let the numeraire be the price of trees, then $\hat{P}(Q)$ is price of consumption.

Since there is nothing dynamic in the choice of a market (note that, we are assuming firms can choose a different market in each period), we can write the problem of a firm as:

$$\pi = \max_{Q} \hat{P}(Q) \Psi^f(Q) z. \tag{24}$$

The first order condition for the optimal choice of Q is

$$\hat{P}'(Q) \Psi^f(Q) + \hat{P}(Q) \Psi^{f'}(Q) = 0, \tag{25}$$

which then determines $\hat{P}(Q)$ as

$$\frac{\hat{P}'(Q)}{\hat{P}(Q)} = -\frac{\Psi^{f'}(Q)}{\Psi^f(Q)}. \tag{26}$$

9 Measure Theory

This section will be a quick review of measure theory to be able to use it in the subsequent sections. In macroeconomics we encounter the problem of aggregation often and it's crucial that we do it in a reasonable way. Measure theory is a tool that tells us when and how we could do so. Let us start with some definitions on sets.

Definition 14 *For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).*

Remark 8 *Note that, in this section we will follow the convention of notations as following*

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition 15 *A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if*

1. *$S, \emptyset \in \mathcal{S}$;*
2. *$A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements); and,*
3. *for $\{B_i\}_{i\in\mathbb{N}}$, $B_i \in \mathcal{S}$ for all i implies $\bigcap_{i\in\mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections).*

Example 1

1. *The power set of S (i.e. all the possible subsets of a set S), is a σ-algebra in S.*
2. *$\{\emptyset, S\}$ is a σ-algebra in S.*
3. *$\left\{\emptyset, S, S_{1/2}, S_{2/2}\right\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.*
4. *If $S = [0, 1]$, then*

$$\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$$

is not *a σ-algebra in S. But*

$$\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$$

is a σ-algebra in S.

Why do we need this σ-algebra? The answer is it defines which sets may be considered as "events": things that could happen. Elements not in it may have no properly defined measure. Basically, σ-algebra is the "patch" that lets us avoid some pathological behaviors of mathematics, namely non-measurable sets. We are now ready to define a measure.

Definition 16 *Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies*

1. $x(\emptyset) = 0$;
2. $B_1, B_2 \in \mathcal{S}$ *and* $B_1 \cap B_2 = \emptyset$ *implies* $x(B_1 \cup B_2) = x(B_1) + x(B_2)$ *(additivity); and,*
3. $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ *and* $B_i \cap B_j = \emptyset$*, for all* $i \neq j$*, implies* $x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity).*[9]

Put simply, a measure is just a way to assign each possible "event" a non-negative real number. A set S, a σ-algebra in it, $\mathcal{S}$, and a measure on $\mathcal{S}$, define a measure space, $(S, \mathcal{S}, x)$.

Definition 17 *Borel σ-algebra is a σ-algebra generated by the family of all open sets (generated by a topology).*

Since a Borel σ-algebra contains all the subsets generated by the intervals, you can recognize any subset of a set using Borel σ-algebra. In other words, Borel σ-algebra corresponds to complete information.

Definition 18 *A probability (measure) is a measure with the property that $x(S) = 1$.*

Definition 19 *Given a measure space $(S, \mathcal{S}, x)$, a function $f : S \to \mathbb{R}$ is measurable (with respect to the measure space) if, for all $a \in \mathbb{R}$, we have*

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

One way to interpret a σ-algebra is that it describes the information available based on observations; a structure to organize information, and how fine are the information that we receive. Suppose that S is

[9] Countable additivity means that the measure of the union of countable disjoint sets is the sum of the measure of these sets.

comprised of possible outcomes of a dice throw. If you have no information regarding the outcome of the dice, the only possible sets in your σ-algebra can be $\emptyset$ and S. If you know that the number is even, then the smallest σ-algebra given that information is $\mathcal{S} = \{\emptyset, \{2,4,6\}, \{1,3,5\}, S\}$. Measurability has a similar interpretation. A function is measurable with respect to a σ-algebra $\mathcal{S}$, if it can be evaluated under the current measure space $(S, \mathcal{S}, x)$.

Example 2 *Suppose $S = \{1,2,3,4,5,6\}$. Consider a function f which maps the element 6 to a number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1,2,3\}, \{4,5,6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1,2,3,4,5\}$. But this set is not in $\mathcal{S}$.*

We can also generalize Markov transition matrix to any measurable space. This is what we do next.

Definition 20 *A function $Q : \mathcal{S} \times S \to [0,1]$ is a transition probability if*

1. *$Q(\cdot, s)$ is a probability measure for all $s \in S$; and,*
2. *$Q(B, \cdot)$ is a measurable function for all $B \in \mathcal{S}$.*

Intuitively, given $B \in \mathcal{S}$ and $s \in S$, $Q(B, s)$ gives the probability of being in set B tomorrow, given that the state is s today. Consider the following example: a *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on the set $S = \{1,2,3\}$, with the σ-algebra $\mathcal{S} = P(S)$ (where $P(S)$ is the power set of S). If Γ_{ij} denotes the probability of state j happening, given a present state i, then

$$Q(\{1,2\}, 3) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 .$$

As another example, suppose we are given a measure x on $\mathcal{S}$; x_i gives us the fraction of type i, for $i \in S$. Given the previous transition function, we can calculate the fraction of types tomorrow using the following formulas:

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words

$$\mathbf{x}' = \Gamma^T \mathbf{x},$$

where $\mathbf{x}^T = (x_1, x_2, x_3)$.

To extend this idea to a general case with a general transition function, we define an *updating operator* as $T(x, Q)$, which is a measure on S with respect to the σ-algebra $\mathcal{S}$, such that

$$\begin{aligned} x'(B) =& T(x, Q)(B) \\ =& \int_S Q(B, s)\, x(ds), \quad \forall B \in \mathcal{S}. \end{aligned}$$

A stationary distribution is a fixed point of T, that is x^* so that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

We know that, if Q has nice properties,[10] then a unique stationary distribution exists (for example, we discard *flipping* from one state to another), and

$$x^* = \lim_{n\to\infty} T^n(x_0, Q),$$

for any x_0 in the space of measures on $\mathcal{S}$.

[10] See Chapter 11 in **?**.

Exercise 29 *Consider unemployment in a very simple economy (we have an exogenous transition matrix). There are two states; employed and unemployed. The transition matrix is given by*

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

10 Industry Equilibrium

10.1 Preliminaries

Now we are going to study a type of models initiated by **?**. We will abandon the general equilibrium framework from the previous section to study the dynamics of distribution of firms in a partial equilibrium environment.

To motivate things, let's start with the problem of a single firm that produces a good using labor input according to a technology described by the production function f. Let us assume that this function is increasing, strictly concave, with $f(0) = 0$. A firm that hires n units of labor is able to produce $sf(n)$, where s is a productivity parameter. Markets are competitive, in the sense that a firm takes prices as given and chooses n in order to solve

$$\pi(s,p) = \max_{n \geq 0} \{psf(n) - wn\}.$$

The first order condition implies that in the optimum, n^*,

$$psf_n(n^*) = w.$$

Let us denote the solution to this problem as a function $n^*(s,p)$.[11] Given the above assumptions, n^*

[11] As we declared in advance, this is a partial equilibrium analysis. Hence, we ignore the dependence of the solution on

is an increasing function of s (i.e. more productive firms have more workers), as well as p.

Suppose now there is a mass of firms in the industry, each associated with a productivity parameter $s \in S \subset \mathbb{R}_+$, where $S := [\underline{s}, \bar{s}]$. Let $\mathcal{S}$ denote a σ-algebra on S (Borel σ-algebra for instance). Let x be a measure defined over the space $(S, \mathcal{S})$ that describes the cross sectional distribution of productivity among firms. Then, for any $B \subset S$ with $B \in \mathcal{S}$, $x(B)$ is the mass of firms having productivities in S.

We will use x to define statistics of the industry. For example, at this point, it is convenient to define the aggregate supply of the industry. Since individual supply is just $sf(n^*(s,p))$, the aggregate supply can be written as[12]

$$Y^S(p) = \int_S sf(n^*(s,p))\, x(ds).$$

Observe that Y^S is a function of the price p; for any price, p, $Y^S(p)$ gives us the supply in this economy.

Exercise 30 *Search Wikipedia for an index of concentration in an industry, and adopt it for our economy.*

Suppose now that the demand of the market is described by some function $Y^D(p)$. Then the equilibrium price, p^*, is determined by the market clearing condition

$$Y^D(p^*) = Y^S(p^*). \tag{27}$$

So far, everything is too simple to be interesting. The ultimate goal here is to understand how the object x is determined by the fundamentals of the industry. Hence, we will be adding tweaks to this basic environment in order to obtain a theory of firms' distribution in a competitive environment. Let's start by allowing firms to die.

w to focus on the determination of p.

[12] S in Y^S stands for supply.

10.2 A Simple Dynamic Environment

Consider now a dynamic environment, in which the situation above repeats every period. Firms discount profits at rate r_t, which is exogenously given. In addition, assume that a single firm, in each period, faces a probability $1-\delta$ of disappearing! We will focus on *stationary equilibria*; i.e. equilibria in which the price of the final output p, the rate of return, r, and the productivity of firm, s, stay constant through time.

Notice first that firm's decision problem is still a static problem; we can easily write the value of an incumbent firm as

$$\begin{aligned} V(s,p) &= \sum_{t=0}^{\infty} \left(\frac{\delta}{1+r}\right)^t \pi(s,p) \\ &= \left(\frac{1+r}{1+r-\delta}\right) \pi(s,p) \end{aligned}$$

Note that we are considering that p is fixed (therefore we can omit it from the expressions above). Observe that every period there is positive mass of firms that die. Therefore, how can this economy be in a stationary equilibrium? To achieve that, we have to assume that there is a constant flow of firms entering the economy in each period, as well.

As before, let x be the measure describing the distribution of firms within the industry. The mass of firms that die is given by $(1-\delta)x(S)$. We will allow these firms to be replaced by new entrants. These entrants draw a productivity parameter s from a probability measure γ.

One might ask what keeps these firms out of the market in the first place? If

$$\pi(s,p) = psf(n^*(s,p)) - wn^*(s,p) > 0,$$

which is the case for the firms operating in the market, then all the (potential) firms with productivities in S would want to enter the market!

We can fix this flaw by assuming that there is a fixed entry cost that each firm must pay in order to operate in the market, denoted by c^E. Moreover, we will assume that the entrant has to pay this cost before learning s. Hence the value of a new entrant is given by the following function:

$$V^E(p) = \int_S V(s,p)\,\gamma(ds) - c^E. \tag{28}$$

Entrants will continue to enter if V^E is greater than 0, and decide not to enter if this value is less than zero. As a result, stationarity occurs when V^E is exactly equal to zero (this is the *free entry* assumption, and we are assuming that there is an infinite number (mass) of prospective firms).

Let's analyze how this environment shapes the distribution of firms in the market. Let x_t be the cross sectional distribution of firms in period t. For any $B \subset S$, portion $1-\delta$ of the firms with productivity $s \in B$ will die, and that will attract some newcomers. Hence, next period's measure of firms on set B will be given by:

$$x_{t+1}(B) = \delta x_t(B) + m\gamma(B).$$

That is, mass m of firms would enter the market in $t+1$, and only fraction $\gamma(B)$ of them will have productivities in the set B. As you might suspect, this relationship must hold for every $B \in \mathcal{S}$. Moreover, since we are interested in stationary equilibria, the previous expression tells us that the cross sectional distribution of firms will be completely determined by γ.

If we let mapping T be defined by

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{29}$$

a stationary distribution of productivity is the fixed point of the mapping T; i.e. x^* with $Tx^* = x^*$, implying:

$$x^*(B;m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}.$$

Now, note that the demand and supply relation in (27) takes the form:

$$y^d\left(p^*\left(m\right)\right) = \int_S sf\left(n^*\left(s,p\right)\right)dx^*\left(s;m\right), \tag{30}$$

whose solution, $p^*\left(m\right)$, is continuous function under regularity conditions stated in **?**.

We have two equations, (28) and (30), and two unknowns, p and m. Thus, we can defined the equilibrium as:

Definition 21 *A stationary distribution for this environment consists of functions p^*, x^*, and m^*, that satisfy:*

1. $y^d\left(p^*\left(m\right)\right) = \int_S sf\left(n^*\left(s,p\right)\right)dx^*\left(s;m\right)$*;*
2. $\int_s V\left(s,p\right)\gamma\left(ds\right) - c^E = 0$*; and,*
3. $x^*\left(B\right) = \delta x^*\left(B\right) + m^*\gamma\left(B\right), \quad \forall B \in \mathcal{S}$*.*

10.3 Introducing Exit Decisions

We want to introduce more (economic) content by making the exit of firms endogenous (a decision of the firm). One way to do so is to assume that the productivity of the firms follow a Markov process governed by a transition function, Γ. This would change the mapping T in Equation (29), as:

$$Tx\left(B\right) = \delta\int_S \Gamma\left(s,B\right)x\left(ds\right) + m\gamma\left(B\right), \quad \forall B \in \mathcal{S}.$$

But, this wouldn't add much economic content to our environment; firms still do not make any (interesting) decision. To change this, let's introduce cost of operation into the model; suppose firms have to pay c^v each period in order to stay in the market. In this case, when s is low, the firm's profit might not cover its cost of operation. So, the firm might decide to leave the market. However, firm has already paid (a sunk cost of) c^E, and, since s changes according to a Markov process, prospects of future profits might deter the firm from quitting. Therefore, negative profit in one period does not imply immediately that the firm's optimal choice is to leave the market.

By adding such a minor change, the solution will have a reservation productivity property under some conditions (to be discussed in the comment below). In words, there will be a minimum productivity, $s^* \in S$, above which it is profitable for the firm to stay in the market.

To see this, note that the value of a firm with productivity $s \in S$ in a period is given by

$$V(s,p) = \max \left\{ 0, \pi(s,p) + \frac{1}{(1+r)} \int_S \Gamma(s, ds') V(s', p) - c^v \right\}.$$

Exercise 31 *Show that the firm's decision takes the form of a reservation productivity strategy, in which, for some $s^* \in S$, $s < s^*$ implies that the firm would leave the market.*

In this case, the transition of the distribution of productivities on S will be:

$$x'(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) x(ds), \quad \forall B \in \mathcal{S}.$$

A stationary distribution of the firms in this economy, x^*, is the fixed point of this equation.

Example 3 *How productive does a firm have to be, to be in the top 10% largest firms in this economy? The answer to this question is the solution to the following equation, $\hat{s}$:*

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1 \ .$$

Then, the fraction of the labor force in the top 10% largest firms in this economy, is

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p) x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p) x^*(ds)}.$$

Exercise 32 *Compute the average growth rate of the smallest one third of the firms. What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next* *period?*

Comment 2 *To see that this will be the case you should prove that the profit before variable cost function $\pi(s,p)$ is increasing in s. Hence the productivity threshold is given by the s^* that satisfies the following condition:*

$$\pi(s^*,p) = c_v$$

for an equilibrium price p. Now instead of considering γ as the probability measure describing the distribution of productivities among entrants, you must consider $\widehat{\gamma}$ defined as follows

$$\widehat{\gamma}(B) = \frac{\gamma(B \cap [s^*, \overline{s}])}{\gamma([s^*, \overline{s}])}$$

for any $B \in \mathcal{S}$.

One might suspect that this is an ad hoc *way to introduce the exit decision. To make things more concrete and easier to compute, we will assume that s is a Markov process. To facilitate the exposition, let's make S finite and assume s has transition matrix Γ. Assume further that Γ is regular enough so that it has a stationary distribution γ^*. For the moment we will not put any additional structure on Γ.*

The operation cost c^v is such that the exit decision is meaningful. Let's analyze the problem from the perspective of the firm's manager. He has now two things to decide. First, he asks himself the question "Should I stay or should I go?". Second, conditional on staying, he has to decide how much labor to hire. Importantly, notice that this second decision is still a static decision. Later, we will introduce adjustment cost that will make this decision a dynamic one.

Let $\phi(s,p)$ be the value of the firm before having decided whether to stay or to go. Let $V(s,p)$ be the value of the firm that has already decided to stay. $V(s,p)$ satisfies

$$V(s,p) = \max_n \left\{ spf(n) - n - c^v + \frac{1}{1+r}\sum_{s' \in S} \Gamma_{ss'}\phi(s',p) \right\}$$

Each morning the firm chooses d in order to solve

$$\phi(s,p) = \max_{d\in\{0,1\}} dV(s,p)$$

Let $d^*(s,p)$ be the optimal decision to this problem. Then $d^*(s,p) = 1$ means that the firm stays in the market. One can alternatively write:

$$\phi(s,p) = \max_{d\in\{0,1\}} d\left[\pi(s,p) - c^v + \frac{1}{1+r}\sum_{s'\in S}\Gamma_{ss'}\phi(s',p)\right]$$

or even

$$\phi(s,p) = \max\left[\pi(s,p) - c^v + \frac{1}{1+r}\sum_{s'\in S}\Gamma_{ss'}\phi(s',p), 0\right] \tag{31}$$

All these are valid. Additionally, one can easily add minor changes to make the exit decision more interesting. For example, things like scrap value or liquidation costs will affect the second argument of the max *operator above, which so far is just zero.*

What about $d^*(s,p)$? Given a price, this decision rule can take only finitely many values. Moreover, if we could ensure that this decision is of the form "stay only if the productivity is high enough and go otherwise" then the rule can be summarized by a unique number $s^* \in S$. Without doubt, that would be very convenient, but we don't have enough structure to ensure that such is the case. Because, although the ordering of s (lower s are ordered before higher s) gives us that the value of s today is bigger than value of smaller s', depending on the Markov chain, on the other hand, the value of productivity level s tomorrow may be lower than the value of s' (note $s' < s$) tomorrow. Therefore we need some additional regularity conditions.

In order to get a cutoff rule for the exit decision, we need to add an assumption about the transition matrix Γ. Let the notation $\Gamma(s)$ indicate the probability distribution over next period state conditional on being on state s today. You can think of it as being just a row of the transition matrix. Take s and $\widehat{s}$. We will say that the matrix Γ displays first order stochastic dominance $(FOSD)$ if $s < \widehat{s}$ implies $\sum_{s'\leq b}\Gamma(s' \mid s) \leq \sum_{s'\leq b}\Gamma(s' \mid \widehat{s})$ for any $b \in S$. It turns out that $FOSD$ is a sufficient condition for *having a cutoff rule. You can prove that by using the same kind of dynamic programming tricks that have been used in a different course for obtaining the reservation wage property in search problems. Try it as an exercise. Also note that this is just a sufficient condition.*

Finally, we need to mention something about potential entrants. Since we will assume that they have to pay the cost c^E before learning their s, they can leave the industry even before producing anything. That requires us to be careful when we describe industry dynamics.
Now the law of motion becomes:

$$x'(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x(ds), \quad \forall B \in \mathcal{S}.$$

10.4 Stationary Equilibrium

Now that we have all the ingredients in the table, let's define the equilibrium formally.

Definition 22 *A stationary equilibrium for this environment consists of a list of functions (ϕ, n^*, d^*), a price p^* and a measure x^* such that*

1. *Given p^*, the functions ϕ, n^*, d^* solve the problem of the incumbent firm*
2. $V^E(p^*) = 0$
3. *For any $B \in \mathcal{S}$ (assuming we have a cut-off rule with s^* is cut-off in stationary distribution)*[13]

[13] If we do not have such cut-off rule we have to define

$$x^*(B) = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{s' \in B\}} \mathbf{1}_{\{d(s', p^*) = 1\}} x^*(ds) + \mu^* \int_S \mathbf{1}_{\{s \in B\}} \mathbf{1}_{\{d(s, p^*) = 1\}} \gamma(ds)$$

where

$$\mu^* = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{d(s', p^*) = 0\}} x^*(ds)$$

$$x^{*}\left(B\right)=m\gamma\left(B\cap\left[s^{*},\bar{s}\right]\right)+\int_{s^{*}}^{\bar{s}}\Gamma\left(s,B\cap\left[s^{*},\bar{s}\right]\right)x^{*}\left(ds\right).$$

4. *Market clearing:*

$$Y^{d}(p^{\star})=\int_{s^{\star}}^{\bar{s}}sf(n^{\star}(s,p^{\star}))dx^{\star}(ds)$$

You can think of condition (2) as a "*no money left over the table*" condition, which ensures additional entrants find unprofitable to participate in the industry.

We can use this model to compute interesting statistics. For example the average output of the firm is given by

$$\frac{Y}{N}=\frac{\sum sf(n^{*}(s))x^{*}(ds)}{\sum x^{*}(ds)}$$

Next, suppose that we want to compute the share of output produced by the top 1% of firms. To do this we first need to compute $\widetilde{s}$ such that

$$\frac{\sum_{\widetilde{s}}^{\bar{s}}x^{*}(ds)}{N}=.01$$

where N is the total measure of firms. Then the share output produced by these firms is given by

$$\frac{\sum_{\widetilde{s}}^{\bar{s}}sf(n^{*}(s))x^{*}(ds)}{\sum_{\underline{s}}^{\bar{s}}sf(n^{*}(s))x^{*}(ds)}$$

Suppose now that we want to compute the fraction of firms who are in the top 1% two periods in a row. This is given by

$$\sum_{s\geq\widetilde{s}}\sum_{s'\geq\widetilde{s}}\Gamma_{ss'}x^{*}(ds)$$

We can use this model to compute a variety of other statistics including the Gini coefficient.

10.5 Adjustment Costs

To end with this section it is useful to think about environments in which firm's productive decisions are no longer static. A simple way of introducing dynamics is by adding adjustment costs.

We will consider labor adjustment costs.[14] Consider a firm that enters period t with n_{t-1} units of labor, hired in the previous period. We can consider three specifications for the adjustment costs, due to hiring n_t units of labor in t, $c(n_t, n_{t-1})$:

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha (n_t - n_{t-1})^2$ units of the numeraire good. The cost here depends on the size of the adjustment.
- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha [n_t - (1-\delta) n_{t-1}]^2$ units of the numeraire good, only if $n_t > n_{t-1}$; we can write this as

  $$\mathbf{1}_{\{n_t > n_{t-1}\}} \alpha [n_t - (1-\delta) n_{t-1}]^2 ,$$

  where $\mathbf{1}$ is the indicator function, and δ measures the exogenous attrition of workers in each period.
- *Firing Costs.*

The recursive formulation of the firm's problem would be:

$$V(s, n_-, p) = \max \left\{ 0, \max_{n \geq 0} \; sf(n) - wn - c^v - c(n, n_-) + \frac{1}{(1+r)} \sum_{s' \in S} \Gamma_{ss'} V(s', n, p) \right\}, \quad (32)$$

where c gives the specified cost of adjusting n_- to n. Note that due to limited liability of the firm, the

[14] These costs work pretty much like capital adjustment costs, as one might suspect.

exit value of the firm is 0 and not $-c\left(0, n_{-}\right)$.

Now, a firm is characterized by both its productivity s and labor n_{-} in the previous period. Note that since the production function f has decreasing returns to scale, there exists an amount of labor $\bar{N}$ such that none of the firms hire labor greater than $\bar{N}$. So, $n_{-} \in N := [0, \bar{N}]$. Let $\mathcal{N}$ be a σ-algebra on N. If the labor policy function is $n = g(s, n_{-})$, then the law of motion now becomes:

$$x'\left(B^S, B^N\right) = m\gamma\left(B^S \cap [s^*, \bar{s}]\right)\mathbf{1}_{\{0 \in B^N\}} + \int_{s^*}^{\bar{s}}\int_0^{\bar{N}} \mathbf{1}_{\{g(s,n_-)\in B^N\}}\Gamma\left(s, B^S \cap [s^*, \bar{s}]\right) x\left(ds, dn_-\right),$$

$$\forall B^S \in \mathcal{S}, \quad \forall B^N \in \mathcal{N}.$$

Exercise 33 *Write the first order conditions for the problem in* (32).

Define the recursive competitive equilibrium for this economy.

Exercise 34 *Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to the function c. The firm also pays a cost κ to post vacancies, and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?*

10.6 Non-stationary Equilibrium

Up until now we focus on the *stationary industrial equilibrium*, in which individual firms enter and exit, but the whole distribution of firms stays invariant. A more interesting case is to look at the *non-stationary equilibrium* and examine how the distribution of firms shift across time.

Let's maintain our baseline model (with entry & exit, but no adjustment costs), and think about the economy starting with some (arbitrary) initial distribution of incumbent firms x_0. We can imagine that, without any shocks, the firm distribution would converge to the stationary equilibrium distribution x^* defined in 10.4. And on the transitional path towards the stationary equilibrium, firms would face a sequence of prices $\{p_t\}_{t=0}^{\infty}$. We now feed in shocks. We will maintain that wage is normalized to 1 and prices p_t each period is going to be pinned down by equating the endogenous aggregate supply and ad-hoc aggregate demand which we denote $D(p_t, z_t)$, where z_t is a demand side shock that shifts aggregate demand.

It's important that we make it clear on the nature of this shock. In general, z_t can be *deterministic* or *stochastic*. Deterministic shocks are fully anticipated by agents in the economy. Stochastic shocks, on the other hand, come in a random manner and agents only know the random process that governs them. Solving the model with deterministic shocks are not harder than solving the transitional path of the model with no shocks. But models with stochastic shocks are much harder to solve. We will say for now the z_t shocks are deterministic and thus focus on the notion of perfect foresight equilibrium (PFE).

We are now ready to define the firm's problem. Note now state variables would incorporate both the individual state s (idiosyncratic productivity shock) and aggregate states: z (aggregate demand shock) and x (measure of firms).

$$
\begin{aligned}
V(s, z_t, x_t) &= \max\left\{0, \pi(s, z_t, x_t) + \frac{1}{1+r}\sum_{s'}\Gamma_{ss'}V(s', z_{t+1}, x_{t+1})\right\} \\
s.t. \quad \pi(s, z_t, x_t) &= \max_{n\geq 0} \ p_t(z_t, x_t)sf(n) - wn_t - c^v
\end{aligned} \tag{33}
$$

Note that we can maintain the cutoff property of the decision rule given our regularity conditions. Let's denote the exit cutoff s_t^*. Note that in order to solve the problem, firms need to know the measure of firms. So we need to figure out the law of motion of firm measure. For each $B \in \mathcal{S}$, we should have

$$
x_{t+1}(B) = m_{t+1}\gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}])x_t(ds) \tag{34}
$$

where m_{t+1} is the mass of firms that enter at the beginning of period $t+1$, which is pinned down by the free entry condition

$$
\int V(s, z_t, x_t)\gamma(ds) \leq c^e \tag{35}
$$

with strict equality holds if $m_t > 0$. The distribution of the initial draw γ and entry cost c^e are exogenously given. Finally, the market clearing condition will close the model by pinning down price p_t

$$D(p_t, z_t) = \int_{s_t^*}^{\bar{s}} sp_t f(n^*(s, z_t, x_t))x_t(ds) \tag{36}$$

Exercise 35 *Figure out the time line behind the above formulation of the firm's problem, the law of motion of firm measure, and the free entry condition.*

We can thus define the perfect foresight equilibrium as following

Definition 23 *For a given path of shock realizations $\{z_t\}$ and a initial firm measure x_0, a perfect foresight equilibrium (PFE) for this environment consists of sequences of functions $\{p_t, m_t, s_t^*, x_t\}$, that satisfy:*

1. **Optimality**: *given $\{p_t\}$, $\{s_t^*\}$ solve the firm's problem (33) for each period t.*
2. **Free entry**: *$\int V(s, z_t, x_t)\gamma(ds) \leq c^e$, with strict equality holds if $m_t > 0$.*
3. **Law of motion**: *$x_{t+1}(B) = m_{t+1}\gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}])x_t(ds)$, $\forall B \in \mathcal{S}$.*
4. **Market clearing**: *$D(p_t, z_t) = \int_{s_t^*}^{\bar{s}} sp_t f(n^*(s, z_t, x_t))x_t(ds)$.*

Having figured out the equilibrium of the perfect foresight model, the natural next step is thus to solve the fully stochastic equilibrium. It is actually a much harder one. We will resort to some notion of linearization to achieve that. So we will divert a bit in the next subsection to talk about linear approximation.

10.7 Digression: Linear Approximation

To better understand the linearization, let's look at a very basic growth model and approximate the solution linearly. Consider such a social planner's problem (with full depreciation)

$$\begin{aligned} v(k_t) = \max_{c_t, k_{t+1}} \; & u(c_t) + \beta v(k_{t+1}) \\ \text{s.t. } & c_t + k_{t+1} \leq f(k_t), \;\; \forall t \geq 0 \\ & c_t, k_{t+1} \geq 0, \;\; \forall t \geq 0 \\ & k_0 > 0 \text{ given.} \end{aligned} \tag{37}$$

We can show that $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ is a solution to the above social planner's problem if and only if

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}), \forall t \geq 0 \tag{38}$$

$$c_t + k_{t+1} = f(k_t), \forall t \geq 0 \tag{39}$$

$$\lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{40}$$

Exercise 36 *Prove the above claim.*

We will focus on cases where a steady state k^* exists. Note that the above necessary and sufficient condition give us a second order difference equation system (we can combine the above solution as $\psi(k_t, k_{t+1}, k_{t+2}) = 0$), with exactly two boundary conditions. so the model is totally solvable. The question is how to do that. One obvious option is to find the global solution. For instance, you can guess a k_1, use k_0 and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \ldots$ forward up until some k_T, and adjust k_1 to make sure k_T is close enough to the steady state k^* (this is called forward shooting). Or you can guess a k_{T-1} and do it backward (which is called backward shooting). Or you can guess and adjust the whole path (which is called the extended path method). All these methods will give you a numerical solution starting from an arbitrary k_0 (that's why we call it a *global* solution).

You can see the above process is time consuming. Linearization is a short cut, that can yield good approximation of the solution *locally*, that is, around the neighborhood of some point. Usually, people will do it around the steady state. The idea is simple. We know the true solution is in the form of

$k_{t+1} = g(k_t)$. Let's simply use a linear function to approximate the true solution $g(\cdot)$. Let's say our approximation is $k_{t+1} = \hat{g}(k_t) = a + bk_t$. Then we only need to figure two numbers: a and b. We thus need two conditions. Since we know the steady state is k^*, which means $a + bk^* = k^*$, we get one condition for free (remember we approximate around k^*). Where to find the other one?

We can only find it in ψ and our criteria is that we are going to choose b such that the slope of $\hat{g}$ exactly matches the slope of true decision rule g *at the steady state* k^*. So we take a first order Taylor expansion of $\psi[k, g(k), g(g(k))]$ around k^* and obtain

$$\psi[k, g(k), g(g(k))] \approx \psi(k^*, k^*, k^*) + \psi_k(k^*, k^*, k^*)(k - k^*) \tag{41}$$

We know $\psi[k, g(k), g(g(k))] = 0$, and k is in the neighborhood of k^*, so it must be

$$\psi_k(k^*, k^*, k^*) = \psi_1^* + \psi_2^* g'(k^*) + \psi_3^* g'(k^*) g'(k^*) = 0 \tag{42}$$

Solve this equation gives us $g'(k^*)$ which is exactly what we need (note ψ_1, ψ_2, and ψ_3 may also involve $g'(k^*)$). We can then let $b = g'(k^*)$ and use $\hat{g}(k_t) = a + bk_t$ to approximate the solution near the steady state.

Comment 3 *In practice, it's messy to do the total derivative as above. A cleaner way is to linearize the system directly with k_t, k_{t+1}, k_{t+2}, and then solve the linear system using whatever method you like. Usually, we cast it on a state space and solve it using matrix algebra (here it helps to know some econometrics).*

Exercise 37 *Suppose $f(k_t) = k_t^\alpha$, $u(c_t) = \ln c_t$. Verify that the solution to the social planner's problem is $k_{t+1} = \alpha\beta k_t^\alpha$. Get the linearized solution around the steady state and compare it with the closed form solution. How precise is the linear approximation?*

Exercise 38 *Extend the linearization to the case where we have stochastic productivity shocks z_t.*