Econ 704 Macroeconomic Theory Spring 2018*

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*This is the evolution of class notes by many students over the years, both from Penn and Minnesota including Makoto Nakajima (2002), Vivian Zhanwei Yue (2002-3), Ahu Gemici (2003-4), Kagan (Omer Parmaksiz) (2004-5), Thanasis Geromichalos (2005-6), Se Kyu Choi (2006-7), Serdar Ozkan (2007), Ali Shourideh (2008), Manuel Macera (2009), Tayyar Buyukbasaran (2010), Bernabe Lopez-Martin (2011), Rishabh Kirpalani (2012), Zhifeng Cai (2013), Alexandra (Sasha) Solovyeva (2014), Keyvan Eslami (2015), Sumedh Ambokar (2016), Ömer Faruk Koru (2017), Jinfeng Luo (2018), and Ricardo Marto (2019).

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1 Introduction

A model is an artificial economy. The description of a model's environment may include specifying the agents' preferences and endowment, technology available, information structure as well as property rights. The Neoclassical Growth Model is one of the workhorses of modern macroeconomics because it delivers some fundamental properties of industrialized economies, summarized by, among others, Kaldor (1957):

1. Output per capita has grown at a roughly constant rate (2%).
2. The capital-output ratio (where capital is measured using the perpetual inventory method based on past consumption foregone) has remained roughly constant (despite output per capita growth).
3. The capital-labor ratio has grown at a roughly constant rate equal to the growth rate of output.
4. The wage rate has grown at a roughly constant rate equal to the growth rate of output.
5. The real interest rate has been stationary and, during long periods, roughly constant.
6. Labor income as a share of output has remained roughly constant (0.66).
7. Hours worked per capita have been roughly constant.

Equilibrium can be defined as a prediction of what will happen and therefore it is a mapping from environments to outcomes (allocations, prices, etc.). One equilibrium concept that we will deal with is Competitive Equilibrium.[1] Characterizing the equilibrium, however, usually involves finding solutions to a system of an infinite number of equations. There are generally two ways of getting around this. First, invoke the welfare theorem to solve for the allocation and then find the equilibrium prices associated with it. This may sometimes not work due to, say, the presence of externalities. The second way is to resort to dynamic programming and study a Recursive Competitive equilibrium, in which equilibrium objects are functions instead of variables.

[1] Arrow-Debreu or Valuation Equilibrium.

2 Review: Neoclassical Growth Model

We review briefly the basic neoclassical growth model.

2.1 The Neoclassical Growth Model (Without Uncertainty)

The commodity space is

$$\mathcal{L} = \{(l_1, l_2, l_3) : l_i = (l_{it})_{t=0}^{\infty}\ l_{it} \in \mathbb{R},\ \sup_t |l_{it}| < \infty,\ i = 1, 2, 3\}.$$

The consumption possibility set is

$$X(\overline{k}_0) = \{x \in \mathcal{L} :\ \exists (c_t, k_{t+1})_{t=0}^{\infty} \text{ s.th. } \forall t = 0, 1, \ldots$$
$$c_t, k_{t+1} \geq 0,\ x_{1t} + (1-\delta)k_t = c_t + k_{t+1},\ -k_t \leq x_{2t} \leq 0,\ -1 \leq x_{3t} \leq 0,\ k_0 = \overline{k}_0\}.$$

The production possibility set is $Y = \prod_t Y_t$, where

$$Y_t = \{(y_{1t}, y_{2t}, y_{3t}) \in \mathbb{R}^3 :\ 0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})\}.$$

Definition 1 *An Arrow-Debreu equilibrium is $(x^*, y^*) \in X \times Y$, and a continuous linear functional ν^* such that*

1. $x^* \in \arg\max_{x \in X, \nu^*(x) \leq 0} \sum_{t=0}^{\infty} \beta^t u(c_t(x), -x_{3t})$,
2. $y^* \in \arg\max_{y \in Y} \nu^*(y)$,
3. *and* $x^* = y^*$.

Note that in this definition we have added leisure. Now, let's look at the one-sector growth model's

Social Planner's Problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, -x_{3t}) \qquad (SPP)$$

s.t.

$$c_t + k_{t+1} - (1-\delta)k_t = x_{1t}$$

$$-k_t \le x_{2t} \le 0$$

$$-1 \le x_{3t} \le 0$$

$$0 \le y_{1t} \le F(-y_{2t}, -y_{3t})$$

$$x = y$$

k_0 given.

Suppose we know that a solution in sequence form exists for (SPP) and is unique.

Exercise 1 *Clearly stating sufficient assumptions on utility and production function, show that (SPP) has a unique solution.*

Two important theorems show the relationship between CE allocations and Pareto optimal allocations:

Theorem 1 (FWT) *Suppose that for all $x \in X$ there exists a sequence $(x_k)_{k=0}^{\infty}$, such that for all $k \ge 0$, $x_k \in X$ and $U(x_k) > U(x)$. If (x^*, y^*, ν^*) is an Arrow-Debreu equilibrium, then (x^*, y^*) is Pareto efficient allocation.*

Theorem 2 (SWT) *If X is convex, preferences are convex, U is continuous, Y is convex and has an interior point, then for any Pareto efficient allocation (x^*, y^*) there exists a continuous linear functional ν such that (x^*, y^*, ν) is a quasiequilibrium, that is:*
(i) for all $x \in X$ such that $U(x) \ge U(x^*)$ it implies $\nu(x) \ge \nu(x^*)$;
(ii) for all $y \in Y$, $\nu(y) \le \nu(y^*)$.

Note that at the very basis of the CE definition and welfare theorems there is an implicit assumption of perfect commitment and perfect enforcement. Note also that the FWT implicitly assumes there is no externality or public goods (achieves this implicit assumption by defining a consumer's utility function only on his own consumption set but no other points in the commodity space). The Greenwald-Stiglitz theorem establishes the Pareto inefficiency of market economies with imperfect information and incomplete markets.

From the First Welfare Theorem, we know that if a Competitive Equilibrium exists, it is Pareto Optimal. Moreover, if the assumptions of the Second Welfare Theorem are satisfied and if the SPP has a unique solution, then the competitive equilibrium allocation is unique and is the same as the PO allocation. Prices can be constructed using this allocation and first-order conditions.

Exercise 2 *Show that*

$$\frac{v_{2t}}{v_{1t}} = F_k(k_t, l_t) \text{ and } \frac{v_{3t}}{v_{1t}} = F_l(k_t, l_t).$$

One shortcoming of the AD equilibrium is that all trade occurs at the beginning of time. This assumption is unrealistic. Modern economics is based on sequential markets. Therefore, we define another equilibrium concept, Sequential Markets Equilibrium (SME). We can easily show that SME is equivalent to ADE by introducing AD securities. All of our results still hold and SME is the right problem to solve.

Exercise 3 *Define a Sequential Markets Equilibrium (SME) for this economy. Prove that the objects we get from the AD equilibrium satisfy SME conditions and that the converse is also true. We should first show that a CE exists and therefore coincides with the unique solution of (SPP).*

Note that the (SPP) problem is hard to solve, since we are dealing with an infinite number of choice variables. We have already established the fact that this SPP problem is equivalent to the following dynamic problem (removing leisure from now on), which is easier to solve:

$$\begin{aligned} v(k) = \max_{c,k'} \quad & u(c) + \beta v(k') \qquad (RSPP) \\ \text{s.t. } \; & c + k' = f(k). \end{aligned}$$

2.2 A Comment on the Welfare Theorems

Situations in which the welfare theorems would not hold include externalities, public goods, situations in which agents are not price-takers (e.g. monopolies), some legal systems or lacking of markets, which rule out certain contracts that appear to be complete or search frictions. What happens in such situations? The solutions to the Social Planner problem and the CE do not coincide, and so we cannot use the theorems we have developed for dynamic programming. As we will see in this course, we can work with Recursive Competitive Equilibria. In general, we can prove that the solution to the RCE coincides with the one derived from the SME, but not the other way around (for example when we have multiple equilibria). However, in all the models we see in this course, this equivalence will hold.

3 Recursive Competitive Equilibrium

3.1 A Simple Example

We have so far established the equivalence between the allocation of the SP problem, which gives the unique Pareto optimal allocation, and the allocations of the AD equilibrium and the SME. We can now solve for the very complicated equilibrium allocation by solving the relatively easier Dynamic Programming problem of the social planner. One handicap of this approach is that in many environments the equilibrium is not Pareto Optimal and hence not a solution of a social planner's problem (e.g. when taxes are distortionary or when externalities are present). Therefore, the recursive formulation of the problem would not be the right problem to solve. In some of these situations we can still write the problem in sequence form. However, we would lose the powerful computational techniques of dynamic programming. In order to resolve this issue we will define the Recursive Competitive Equilibrium equivalent to SME that we can always solve for.

In order to write the household problem recursively, we need to use equilibrium conditions so that the household knows what prices are, in particular, as a function of some economy-wide aggregate state variables. Let aggregate capital be K and aggregate labor $L = 1$. Then from solving the firm's problem, factor prices are given by $w(K) = F_n(K, 1)$ and $R(K) = F_k(K, 1)$. Therefore, since households take prices as given, they need to know aggregate capital. A household who is deciding about how much to consume and how much to work has to know the whole sequence of future prices in order to make her decision and that means that she needs to know the path of aggregate capital. Therefore, if she believes that aggregate capital changes according to the mapping G such that $K' = G(K)$, then knowing aggregate capital today she would be able to project the path of aggregate capital into the future and thus the path for prices. So, we can write the household problem given function $G(\cdot)$ as follows:

$$\begin{aligned} \Omega(K, a; G) = \max_{c,a'} \quad u(c) &\quad + \quad \beta\Omega(K', a'; G) \qquad\qquad (RCE) \\ \text{s.t. } c + a' &= w(K) + R(K)a \\ K' &= G(K), \\ c &\geq 0 \end{aligned}$$

The above problem is the problem of a household that sees K in the economy, has a belief G about its evoluation, and carries a units of assets from past. The solution of this problem yields policy functions $c(K, a; G)$ for consumption and $g(K, a; G)$ for next period asset holdings, as well as a value function $\Omega(K, a; G)$. The price functions $w(K), R(K)$ are obtained from the firm's FOCs (below).

$$\begin{aligned} u_c(c(K, a; G)) &= \beta\Omega_{a'}(G(K), g(K, a; G); G) \\ \Omega_a(K, a; G) &= R(K)\, u_c(c(K, a; G)) \end{aligned}$$

Now we can define the Recursive Competitive Equilibrium.

Definition 2 *A Recursive Competitive Equilibrium with arbitrary expectations G is a set of functions Ω, $g : \mathcal{K} \times \mathcal{A} \to \mathbb{R}$, and R, w, $H : \mathcal{K} \to \mathbb{R}_+$ such that:*[2]

[2] Note that we could add the policy function for consumption $c(K, a; G)$.

1. *Given G, w, R, Ω, g solves the household problem in (RCE),*

2. $K' = H(K; G) = g(K, K; G)$ *(representative agent condition),*

3. $w(K) = F_n(K, 1)$, *and*

4. $R(K) = F_k(K, 1)$.

Note that G are some arbitrary expectations and do not have to necessarily be rational. Next, we define another notion of equilibrium where the expectations of the households are consistent with what happens in the economy:

Definition 3 *A Rational Expectations Recursive Competitive Equilibrium (REE) is a set of functions Ω, g, R, w, G^*, such that:*

1. *Given w, R, $\Omega(K, a; G^*), g(K, a; G^*)$ solves HH problem in (RCE),*

2. $K' = G^*(K) = g(K, K; G^*)$,

3. $w(K) = F_n(K, 1)$, *and*

4. $R(K) = F_k(K, 1)$.

What this means is that in a REE, households optimize given what they believe is going to happen in the future and what happens in the aggregate is consistent with the household's decision. The proof that every REE can be used to construct a SME is left as an exercise. The reverse turns out not to be true. Notice that in a REE, function G^* projects next period's aggregate capital. In fact, if we construct an equilibrium path based on a REE, once a level of aggreagte capital is reached in some period, then next period aggreagte capital is uniquely pinned down by the transition function G^*. If we have multiplicity of SME, this would imply that we cannot construct the function G^* since one value of capital today could imply more than one value of capital tomorrow. From now on, we will focus on REE unless otherwise stated.

Remark 1 *Note that unless otherwise stated, we will assume that the capital depreciation rate δ is 1, with the firm's profits given by $F(K,1)-\delta K - r(K)K - w(K)$. $R(K)$ is the gross rate of return on capital, which is given by $F_k(K,1)+1-\delta$. The net rate of return on capital is $r(K)=F_k(K,1)-\delta$.*

3.2 The Envelope Theorem and the Functional Euler equation

To solve for the RCE and, in particular, to derive the household's optimality conditions we use envelope theorem. This method is valid because of time consistency of consumption choice.

Take the household's problem given by

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad u(c) &\quad + \quad \beta V(K',a') \\ \text{s.t.} \quad c + a' &\quad = \quad w(K) + R(K)a \\ K' &\quad = \quad G(K) \\ c &\quad \geq \quad 0 \end{aligned}$$

with decision rules for consumption and next period asset holdings given by $c=c(K,a)$ and $a'=g(K,a)$.

By taking the first-order conditions (assuming an interior solution since u is well behaved), we get:

$$-u_c(c) + \beta V_{a'}(K',a') = 0,$$

which evaluated at the optimum is

$$-u_c\left(w(K)+R(K)a-g(K,a)\right)+\beta V_{a'}\left(G(K),g(K,a)\right)=0 \tag{1}$$

The problem with solving the functional Euler equation is that $V_{a'}$ is not known. However, we can write the value function as a function of current states and differentiate both sides with respect to a.[3] Since the Euler equation holds for all (a, K), we have

$$V(K,a) = u(w(K) + R(K)a - g(K,a)) + \beta V(G(K), g(K,a)) \tag{2}$$

and using the implicit function theorem we can get its derivative with respect to a:

$$\begin{aligned} V_a(K,a) &= u_c(w(K) + R(K)a - g(K,a))R(K) + \\ &\frac{\partial g(K,a)}{\partial a}\left[-u_c(w(K) + R(K)a - g(K,a)) + \beta V_{a'}(G(K), g(K,a))\right] \end{aligned} \tag{3}$$

The term in square brackets in the right hand side is the first-order condition (1) and hence it is zero. So equation (3) simplifies to $V_a(K,a) = u_c(w(K)+R(K)a-g(K,a))R(K)$. Note, however, that we need $V_{a'}(G(K), g(K,a))$ to find the optimal asset holdings allocation. We would need to follow the same procedure for $V(G(K), g(K,a))$, but since equation 1 holds for all (a, K) next period's Euler equation is $u_{c'}(w(G(K))+R(G(K))g(K,a)-g(G(k), g(K,a))) = \beta V_{a''}\Big[G(G(K)), g(G(K), g(K,a))\Big]$. This in turn implies that $V_{a'}(G(K), g(K,a)) = u_{c'}(w(G(K))+R(G(K))g(K,a)-g(G(k), g(K,a)))\ R(G(K))$.

Finally, we can replace that in equation (1) and get the functional Euler equation

$$u_c\left(w(K) + R(K)a - g(K,a)\right) - \beta u_{c'}(w(G(K))+R(G(K))g(K,a)-g(G(k), g(K,a)))\ R(G(K)) = 0 \tag{4}$$

To illustrate this point, consider an individual who wants to loose weight and decides whether to start diet or not. However, he would rather postpone diet for tomorrow and prefer to eat well today. Let 1 denotes that he obeys the diet restrictions and 0 otherwise. Let his preference ordering be given by:

1. $(0, 1, 1, 1...)$
2. $(1, 1, 1, 1...)$

[3] Under some assumptions, V is differentiable. See p. 121 of Prof. Krueger's notes for details.

3. $(0, 0, 0, 0, ...)$

Even though he promises himself that he will start diet tomorrow and chooses to eat well today, tomorrow he will face the same problem. So he will choose the same option again tomorrow. He will thus never start diet and will end up with his least preferred option: $(0, 0, 0, 0, ...)$.

However, in our model that is not what happens. Agents' preferences are time consistent, so what an individual promises today has to be optimal for her tomorrow as well. And that is why we can use the envelope theorem.

3.3 Economies with Government Expenditures

3.3.1 Lump-Sum Tax

The government levies each period T units of goods in a lump sum way and spends it in a public good, say, medals. Assume however that consumers do not care about medals. The household's problem becomes:

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & u(c) \quad + \quad \beta V(K',a') \\ \text{s.t.} \quad c + a' \quad & = \quad w(K) + R(K)a - T \\ K' \quad & = \quad G(K) \\ c \quad & \geq \quad 0 \end{aligned}$$

Let the solution of this problem be given by policy function $g_a(K, a; M, T)$ and value function $V(K, a; M, T)$. The equilibrium can be characterized by $G^*(K; M, T) = g_a(K, K; G^*, M, T)$ and $M^* = T$ (the government budget constraint is balanced period by period). We will write a complete definition of equilibrium for a version with government debt (below).

Note that the equilibrium will be optimal. But if households cared about medals, then the equilibrium would not be optimal in general.

Exercise 4 *Define* $\hat{f}(K,1) = f(K,1) - M$ *for the planner. Show that the equilibrium is optimal when consumers do not care about medals.*

3.3.2 Labor Income Tax

We have an economy in which the government levies tax on labor in order to purchase medals. Medals are goods which provide utility to the consumers.

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad & u(c,M) \quad + \quad \beta V(K',a') \\ \text{s.t.} \quad c + a' \quad & = \quad (1-\tau(K))\, w(K) + R(K)a \\ K' \quad & = \quad G(K) \\ c \quad & \geq \quad 0 \end{aligned}$$

with $M = \tau(K) w\,(K)$.

Since leisure is not valued, the labor decision is trivial. Hence, there is no distortion due to taxes and CE is Pareto optimal. This will also hold when medals do not provide any utility to the consumers.

Exercise 5 *Is there any change in the above implications of optimality if the tax rate is a function of aggregate capital?*

Exercise 6 *Suppose medals do not provide utility to agents but leisure does. Is the CE optimal now? Is it distorted? What if medals also provide utility?*

3.3.3 Capital Income Tax

Now let us look at an economy in which the government levies tax on capital in order to purchase medals. Medals are goods which provide utility to the consumers.

$$\begin{aligned} V(K,a) = \max_{c,a'} \quad u(c,M) \quad &+ \quad \beta V(K',a') \\ \text{s.t.} \quad c + a' \quad &= \quad w(K) + a\left[1 + r(K)\left(1 - \tau(K)\right)\right] \\ K' \quad &= \quad G(K) \\ c \quad &\geq \quad 0 \end{aligned}$$

with $M = \tau(K) r(K) K$ and $R(K) = 1 + r(K)$. Now, the First Welfare Theorem is no longer applicable and the CE will therefore not be Pareto optimal anymore (if $\tau(K) > 0$ there will be a wedge, and the efficiency conditions will not be satisfied).

Exercise 7 *Derive the first order conditions in the above problem to see the wedge introduced by taxes.*

3.3.4 Taxes and Debt

Assume that the government can now issue debt and use taxes to finance its expenditures. Also assume that agents derive utility from these government expenditures.

A government policy consists of capital taxes, spending (medals) as well as bond issuance. When the aggregate states are K and B, as you will see why, then a government policy (in a recursive world) is

$$\tau(K,B), \; M(K,B) \text{ and } B'(K,B).$$

For now, we shall assume these values are chosen so that the equilibrium exists. In this environment, debt issued is relevant for the household because it permits him to correctly infer the amount of taxes. Therefore the household needs to form expectations about the future level of debt from the government.

The government budget constraint now satisfies (with taxes on labor income):

$$B + M(K,B) = \tau(K,B)R(K)K + q(K,B)B'(K,B)$$

Notice that the household does not care about the composition of his portfolio as long as assets have the same rate of return, which is true because of the no arbitrage condition.

The problem of a household with assets a is given by:

$$\begin{aligned} V(K,B,a) \quad = \quad \max_{c,a'} \quad & u(c, M(K,B)) \quad + \quad \beta V(K',B',a') \\ \text{s.t.} \quad c + a' \;=\; & w(K) + aR(K)(1-\tau(K,B)) \\ K' \;=\; & G(K,B) \\ B' \;=\; & H(K,B) \\ c \;\geq\; & 0 \end{aligned}$$

Let $g(K,B,a)$ be the policy function associated with this problem. Then, we can define a RCE as follows.

Definition 4 *A* Rational Expectations Recursive Competitive Equilibrium, *given policies $M(K,B)$ and $\tau(K,B)$, is a set of functions V, g, G, H, w, and R, such that*

1. *Given w and R, V and g solve the household's problem,*

2. *Factor prices are paid their marginal productivities*

   $$w(K) = F_2(K,1) \text{ and } R(K) = F_1(K,1),$$

3. *Household wealth = Aggregate wealth*

   $$g(K,B,K + q(K^-,B^-)B) = G(K,B) + q(K,B)H(K,B),$$

4. *No arbitrage condition*

$$\frac{1}{q(K,B)} = [1 - \tau(G(K,B), H(K,B))]\, R(G(K))$$

5. *Government's budget constraint holds*

$$B + M(K,B) = \tau(K,B)R(K)K + q(K,B)H(K,B)$$

6. *Government debt is bounded; i.e.* $\exists$ *some* $\bar{B}$ *such that for all* $K \in \left[0, \widetilde{k}\right)$ *and* $B \leq \bar{B}$, $H\left(K,B\right) \leq \bar{B}$.