# Econ 704 Macroeconomic Theory Spring 2020*

José-Víctor Ríos-Rull

University of Pennsylvania

CAERP

March 25, 2020

*This is the evolution of class notes by many students over the years, both from Penn and Minnesota including Makoto Nakajima (2002), Vivian Zhanwei Yue (2002-3), Ahu Gemici (2003-4), Kagan (Omer Parmaksiz) (2004-5), Thanasis Geromichalos (2005-6), Se Kyu Choi (2006-7), Serdar Ozkan (2007), Ali Shourideh (2008), Manuel Macera (2009), Tayyar Buyukbasaran (2010), Bernabe Lopez-Martin (2011), Rishabh Kirpalani (2012), Zhifeng Cai (2013), Alexandra (Sasha) Solovyeva (2014), Keyvan Eslami (2015), Sumedh Ambokar (2016), Ömer Faruk Koru (2017), Jinfeng Luo (2018), and Ricardo Marto (2019 and 2020).

# Contents


**1 Introduction** **3**

**2 Review: Neoclassical Growth Model** **4**

2.1 The Neoclassical Growth Model (Without Uncertainty) . . . . . . . . . . . . . . . . . . 4

2.2 A Comment on the Welfare Theorems . . . . . . . . . . . . . . . . . . . . . . . . . . 7

**3 Recursive Competitive Equilibrium** **7**

3.1 A Simple Example . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

# 1 Introduction

A model is an artificial economy used to understand economic phenomena. The description of a model's environment includes specifying agents' preferences and endowments, technology available, information structure as well as property rights. One such example is the Neoclassical Growth Model. It is one of the workhorse frameworks of modern macroeconomics because it delivers some fundamental properties that are characteristics of industrialized economies. Kaldor (1957) summarizes these six stylized facts (the seventh was added later on):

1. Output per capita ($Y/L$) has grown at a roughly constant rate (of 2%).
2. The capital-output ratio ($K/Y$, where capital is measured using the perpetual inventory method) has remained roughly constant (despite output per capita growth).
3. The capital-labor ratio ($K/L$) has grown at a roughly constant rate equal to the growth rate of output.
4. Labor income as a share of output ($WL/Y$) has remained roughly constant (0.66).
5. The wage rate has grown at a roughly constant rate equal to the growth rate of output.
6. The real interest rate has been stationary and, during long periods, roughly constant.
7. Hours worked per capita have been roughly constant.

A model is not complete without the notion of an equilibrium concept. Equilibrium can be defined as a prediction of what will happen in the economy, i.e. a mapping from environments to outcomes (allocations, prices, etc.). One equilibrium concept that we will deal with during the course is the Competitive Equilibrium (CE). Characterizing the equilibrium usually involves finding solutions to a system of an infinite number of equations.[1] There are generally two ways of getting around this challenge. The first is to invoke the first welfare theorem to solve for the allocation and then find the equilibrium prices associated with it. However, this may sometimes not work due to, say, the presence of

[1] As in Arrow-Debreu or Valuation Equilibrium.

externalities. The second way is to resort to dynamic programming and study a Recursive Competitive Equilibrium (RCE), in which equilibrium objects are functions instead of sequences. We briefly review the first in the next section and then move on to recursive competitive equilibria.

# 2 Review: Neoclassical Growth Model

We review briefly the basic neoclassical growth model.

## 2.1 The Neoclassical Growth Model (Without Uncertainty)

The commodity space is

$$\mathcal{L} = \{(l_1, l_2, l_3) : l_i = (l_{it})_{t=0}^{\infty}\ l_{it} \in \mathbb{R},\ \sup_t |l_{it}| < \infty,\ i = 1, 2, 3\}.$$

The consumption possibility set is

$$\begin{aligned} X(\bar{k}_0) = \{x \in \mathcal{L} :\ & \exists (c_t, k_{t+1})_{t=0}^{\infty} \text{ s.th. } \forall t = 0, 1, \ldots \\ & c_t, k_{t+1} \geq 0,\ x_{1t} + (1-\delta)k_t = c_t + k_{t+1},\ -k_t \leq x_{2t} \leq 0,\ -1 \leq x_{3t} \leq 0,\ k_0 = \bar{k}_0\}. \end{aligned}$$

The production possibility set is $Y = \prod_t Y_t$, where

$$Y_t = \{(y_{1t}, y_{2t}, y_{3t}) \in \mathbb{R}^3 :\ 0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})\}.$$

**Definition 1** *An Arrow-Debreu equilibrium is $(x^*, y^*) \in X \times Y$, and a continuous linear functional $\nu^*$ such that*

1. $x^* \in \arg\max_{x \in X, \nu^*(x) \leq 0} \sum_{t=0}^{\infty} \beta^t u(c_t(x), -x_{3t})$,
2. $y^* \in \arg\max_{y \in Y} \nu^*(y)$,

3. *and* $x^* = y^*$.

Note that in this definition we have added leisure. Now, let's look at the Social Planner's Problem of the one-sector growth model:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, -x_{3t}) \qquad (SPP)$$

$$\text{s.t.} \qquad c_t + k_{t+1} - (1-\delta)k_t = x_{1t}$$

$$-k_t \leq x_{2t} \leq 0$$

$$-1 \leq x_{3t} \leq 0$$

$$0 \leq y_{1t} \leq F(-y_{2t}, -y_{3t})$$

$$x = y$$

$$k_0 \text{ given.}$$

Suppose we know that a solution in sequence space exists for $(SPP)$ and it is unique.

**Exercise 1** *Clearly stating sufficient assumptions on utility and production function, show that (SPP) has a unique solution.*

Two important theorems show the relationship between CE allocations and Pareto optimal allocations:

**Theorem 1 (FWT)** *Suppose that for all $x \in X$ there exists a sequence $(x_k)_{k=0}^{\infty}$ such that for all $k \geq 0$, $x_k \in X$ and $U(x_k) > U(x)$. If $(x^*, y^*, \nu^*)$ is an Arrow-Debreu equilibrium, then $(x^*, y^*)$ is a Pareto efficient allocation.*

**Theorem 2 (SWT)** *If $X$ is convex, preferences are convex, $U$ is continuous, $Y$ is convex and has an interior point, then for any Pareto efficient allocation $(x^*, y^*)$ there exists a continuous linear functional $\nu$ such that $(x^*, y^*, \nu)$ is a quasiequilibrium, that is:*
*(i) for all $x \in X$ such that $U(x) \geq U(x^*)$ it implies $\nu(x) \geq \nu(x^*)$;*
*(ii) for all $y \in Y$, $\nu(y) \leq \nu(y^*)$.*

Note that at the very basis of the CE definition and welfare theorems there is an implicit assumption of perfect commitment and perfect enforcement. Note also that the FWT implicitly assumes there is no externality or public goods (it achieves this implicit assumption by defining a consumer's utility function only on his own consumption set but no other points in the commodity space). The Greenwald-Stiglitz (1986) theorem establishes the Pareto inefficiency of market economies with imperfect information and incomplete markets.

From the FWT, we know that if a Competitive Equilibrium exists, it is Pareto Optimal. Moreover, if the assumptions of the SWT are satisfied and if the (SPP) has a unique solution, then the competitive equilibrium allocation is unique and is the same as the Pareto Optimal allocation. Prices can then be constructed using this allocation and first-order conditions.

**Exercise 2** *Show that*

$$\frac{v_{2t}}{v_{1t}} = F_k(k_t, l_t) \text{ and } \frac{v_{3t}}{v_{1t}} = F_l(k_t, l_t).$$

One shortcoming of the Arrow-Debreu Equilibrium (ADE) is that all trade occurs at the beginning of time. This assumption is unrealistic. Modern economics is based on sequential markets. Therefore, we define another equilibrium concept, the Sequential Markets Equilibrium (SME). We can easily show that SME is equivalent to ADE by introducing Arrow-Debreu securities. All of our results still hold and SME is therefore the right problem to solve.

**Exercise 3** *Define a Sequential Markets Equilibrium (SME) for the economy above. Prove that the objects we get from the AD equilibrium satisfy SME conditions and that the converse is also true. We should first show that a CE exists and therefore coincides with the unique solution of (SPP).*

Note that the (SPP) is hard to solve since we are dealing with an infinite number of choice variables. Instead, we can establish that this SPP problem is equivalent to the following dynamic problem (removing leisure from now on), which is easier to solve:

$$
\begin{aligned}
v(k) = \max_{c,k'} \quad & u(c) + \beta v(k') \qquad (RSPP) \\
\text{s.t.} \qquad & c + k' = f(k).
\end{aligned}
$$

### 2.2 A Comment on the Welfare Theorems

Situations in which the welfare theorems would not hold include externalities, public goods, situations in which agents are not price-takers (e.g. monopolies), some legal systems, when markets are missing, which could rule out certain contracts that appear to be complete, or search frictions. What happens in such situations? The solutions to the Social Planner problem and the CE do not coincide, and so we cannot use the welfare theorems we have developed for dynamic programming. As we will see in this course, we can work with Recursive Competitive Equilibria. In general, we can prove that the solution to the RCE coincides with the one derived from the SME, but not the other way around (for example when we have multiple equilibria). However, in all the models we will see in this course, this equivalence will hold.

## 3 Recursive Competitive Equilibrium

### 3.1 A Simple Example

We have so far established the equivalence between the allocation of the SP problem, which gives the unique Pareto optimal allocation, and the allocations of the AD equilibrium and the SME. We can now solve for the very complicated equilibrium allocation by solving the relatively easier Dynamic Programming problem of the social planner. One handicap of this approach is that in many environments the equilibrium is not Pareto Optimal and hence not a solution of a social planner's problem (e.g. when taxes are distortionary or when externalities are present). Therefore, the recursive formulation of the problem (RSPP) would not be the right problem to solve. In some of these situations we can still write the problem in sequence form. However, we would lose the powerful computational techniques of dynamic programming. In order to resolve this issue we will define the Recursive Competitive Equilibrium equivalent to SME that we can always solve for.

We start with the household's problem. In order to write it recursively, we need to use equilibrium conditions that tells the household what prices are, in particular as functions of economy-wide aggregate state variables. Let aggregate capital be $K$ and aggregate labor $N = 1$. Then from solving the firm's problem, factor prices are given by $w(K) = F_n(K, 1)$ and $R(K) = F_k(K, 1)$. Therefore, since households take prices as given, they need to know aggregate capital in order to make their decisions. A household who is choosing how much to consume and how much to work has to know the whole sequence of future prices in order to make her decision. That means that she needs to know the path of aggregate capital. Therefore, if she believes that aggregate capital changes according to the mapping $G$, such that $K' = G(K)$, then knowing aggregate capital today enables her to project the path of aggregate capital into the future and thus the path for prices. So, we can write the household's recursive problem given function $G(\cdot)$ as follows:

$$\begin{aligned} V(K,a;G) = \max_{c,a'} \quad u(c) \quad &+ \quad \beta V(K',a';G) \qquad\qquad (RCE) \\ \text{s.t. } c + a' \quad &= \quad w(K) + R(K)a \\ K' \quad &= \quad G(K), \\ c \quad &\geq \quad 0 \end{aligned}$$

The dynamic programming problem above is for a household that sees $K$ in the economy, has a belief $G$ about its evolution, and carries $a$ units of assets from the past. The price functions $w(K), R(K)$ are obtained from the firm's FOCs. The solution of this problem yields policy functions $c(K, a; G)$ for consumption and $g(K, a; G)$ for next period asset holdings, as well as a value function $V(K, a; G)$, which must satisfy

$$
\begin{aligned}
u_c\left(c(K,a;G)\right) &= \beta V_{a'}\left(G(K), g(K,a;G);G\right) \\
V_a\left(K,a;G\right) &= R(K)\, u_c\left(c(K,a;G)\right)
\end{aligned}
$$

Now we can define the Recursive Competitive Equilibrium.

**Definition 2** *A Recursive Competitive Equilibrium with arbitrary expectations $G$ is a set of functions $V$, $g : \mathcal{K} \times \mathcal{A} \to \mathbb{R}$, and $R$, $w$, $G : \mathcal{K} \to \mathbb{R}_+$ such that:*[2]

1. *Given $G, w, R$, $V$ and $g$ solve the household's problem in (RCE),*
2. *$K' = G(K) = g(K,K;G)$ (representative agent condition),*
3. *$w\left(K\right) = F_n\left(K,1\right)$, and*
4. *$R\left(K\right) = F_k\left(K,1\right)$.*

Note that $G$ represents some arbitrary expectations and do not have to necessarily be rational. Next, we define another notion of equilibrium in which the expectations of the household are consistent with what happens in the economy:

**Definition 3** *A Rational Expectations Recursive Competitive Equilibrium (REE) is a set of functions $V$, $g$, $R$, $w$, $G^*$, such that:*

1. *Given $w, R$, $V(K,a;G^*)$ and $g(K,a;G^*)$ solve the household's problem in (RCE),*
2. *$K' = G^*(K) = g(K,K;G^*)$,*
3. *$w\left(K\right) = F_n\left(K,1\right)$, and*
4. *$R\left(K\right) = F_k\left(K,1\right)$.*

[2] Note that we could add the policy function for consumption $c(K,a;G)$.

What this means is that in a REE households optimize given what they believe is going to happen in the future and what happens in the aggregate is consistent with the households' decision. The proof that every REE can be used to construct a SME is left as an exercise. The reverse turns out not to be true. Notice that in a REE, function $G^*$ projects next period's aggregate capital. If there is a multiplicity of SME, this would imply that we cannot construct such function $G^*$, since one value of capital today could imply more than one value of capital tomorrow, i.e. $G^*$ is not a correspondence. From now on, we will focus on REE unless otherwise stated because it helps us select an equilibrium in case more than one exists.

**Remark 1** *Note that unless otherwise stated, we will assume that the capital depreciation rate $\delta$ is 1, with the firm's profits given by $F(K,1) - (r(K) + \delta)K - w(K)$. $R(K)$ is the gross rate of return on capital, which is given by $R(K) = F_k(K,1) + 1 - \delta$. The net rate of return on capital is $r(K) = F_k(K,1) - \delta$.*